

# Cycle Spaces of Infinite Dimensional Flag Domains

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April, 2016

## Abstract

Let  $G$  be a complex simple direct limit group, specifically  $SL(\infty; \mathbb{C})$ ,  $SO(\infty; \mathbb{C})$  or  $Sp(\infty; \mathbb{C})$ . Let  $\mathcal{F}$  be a (generalized) flag in  $\mathbb{C}^\infty$ . If  $G$  is  $SO(\infty; \mathbb{C})$  or  $Sp(\infty; \mathbb{C})$  we suppose further that  $\mathcal{F}$  is isotropic. Let  $\mathcal{Z}$  denote the corresponding flag manifold; thus  $\mathcal{Z} = G/Q$  where  $Q$  is a parabolic subgroup of  $G$ . In a recent paper [7] we studied real forms  $G_0$  of  $G$  and properties of their orbits on  $\mathcal{Z}$ . Here we concentrate on open  $G_0$ -orbits  $D \subset \mathcal{Z}$ . When  $G_0$  is of hermitian type we work out the complete  $G_0$ -orbit structure of flag manifolds dual to the bounded symmetric domain for  $G_0$ . Then we develop the structure of the corresponding cycle spaces  $\mathcal{M}_D$ . Finally we study the real and quaternionic analogs of these theories. All this extends results from the finite dimensional cases on the structure of hermitian symmetric spaces and cycle spaces (in chronological order: [12], [17], [14], [15], [18], [16], [4], [5], [19], [6]).

## 1 Introduction.

The object of this paper is the study of certain infinite dimensional bounded symmetric domains and the related cycle spaces for open real group orbits on complex flag manifolds. The cycle space theory is well understood in the finite dimensional setting (in chronological order: [12], [17], [14], [15], [18], [16], [4], [5], [19], [6]). Here we initiate its extension to infinite dimensions. Specifically, we look at the action of real reductive direct limit groups,  $G_0$  such as  $SL(\infty; \mathbb{R})$ ,  $SO(\infty, \infty)$ ,  $Sp(\infty, q)$ , or  $Sp(\infty; \mathbb{R})$ , on a class of direct limit complex flag manifolds  $\mathcal{Z} = G/Q$ , where  $G$  is the complexification of  $G_0$ . While the classical finite dimensional setting [12] is the guide, the results in infinite dimensions are much more delicate, and often different. See [7], as indicated below. In fact there are even stringent requirements for the existence of open  $G_0$ -orbits on  $\mathcal{Z}$ . In all cases where  $G_0$  is the group of an hermitian symmetric space we work out a complete structure theory for the cycle spaces of open orbits in our class of flag manifolds. That structure is explicit in terms of the bounded symmetric domains of the  $G_0$ .

In Section 2 we review the basic facts about our class of infinite dimensional complex Lie groups, their construction, their flag manifolds, and their real forms. We note [7] that every  $G_0$ -orbit on  $\mathcal{Z}$  is infinite dimensional, and we describe just when the number of  $G_0$ -orbits on  $\mathcal{Z}$  is finite.

In Section 3 we concentrate on the cases where  $G_0$  is a special linear group or is defined by a bilinear or hermitian form. We then recall foundational results from [7] and describe a notion of nondegeneracy for flags  $\mathcal{F} \in \mathcal{Z}$  (even in the cases  $G_0 = SL(\infty; \mathbb{R})$  and  $G_0 = SL(\infty; \mathbb{H})$ ). We

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\*Research partially supported by a Dickson Emeriti Professorship and by a Simons Foundation grant.

use nondegeneracy to determine which  $G_0$ -orbits are open, and in fact and whether there are any open  $G_0$ -orbits.

In Section 4 we develop a complete structure theory for the finitary infinite dimensional bounded symmetric domains. The results are similar to the classical finite dimensional results, but one has to be careful about the details. We obtain complete extensions of the orbit structure (in particular the boundary structure) from the finite dimensional cases ([8], [11], [12]).

Then in Section 5 we initiate the study of cycle spaces of the open  $G_0$ -orbits on  $\mathcal{Z}$ . We start with the important case of  $G_0 = SU(\infty, q)$ ,  $q \leq \infty$ , using an idea from the finite dimensional setting. We show how that idea leads to a precise description of the cycle space more generally. This is the start of a program to extend results of [3] to infinite dimensions. This study raises many important questions and initiates several promising lines of research. Compare [3].

One could carry out the considerations of Sections 4 and 5 in a more unified way, but there are many small differences of technical detail, so it would not be advantageous.

Finally in Section 6 we carry some of the results of Sections 4 and 5 over to certain real and quaternionic bounded symmetric domains. As noted in [10] this has some physical interest.

This study grew out of a joint project [7] with Ivan Penkov and Mikhail Ignatyev, where we studied real forms  $G_0$  of  $SL(\infty; \mathbb{C})$  and the basic properties of their orbits on flag varieties  $\mathcal{Z}$ . I thank Ivan Penkov for important discussions on early versions of this manuscript, and I thank the referee for the publication version of this paper for useful critical comments.

## 2 Basics.

In this section we review some basic facts about our class of infinite dimensional real and complex Lie groups, complex flag manifolds, and real group orbits.

### 2.1 Direct Limit Groups.

Let  $V$  be a countable dimensional complex vector space and  $E$  a fixed basis of  $V$ . We fix a linear order on  $E$ , specifically by  $\mathbb{N} = \mathbb{Z}^+$ , where  $E = \{e_1, e_2, \dots\}$ . When we come to flags and parabolics we will consider other orders on  $E$ , but we use the given order by  $\mathbb{Z}^+$  to define our groups and our exhaustions of  $V$ .

Let  $V_*$  denote the span of the dual system  $\{e_1^*, e_2^*, \dots\}$ ; we view  $V_*$  as the restricted dual of  $V$ . The group  $GL(V, E)$  is the group of invertible linear transformations on  $V$  that keep fixed all but finitely many elements of  $E$ . It is easy to see that  $GL(V, E)$  depends only on the pair  $(V, V_*)$  as long as  $V_*$  is constructed from  $E$ .

Express the basis  $E$  as an increasing union  $E = \bigcup E_n$  of finite subsets. That exhausts  $V$  by finite dimensional subspaces  $V_n = \text{Span}\{E_n\}$ ,  $V = \varinjlim V_n$ , and thus expresses  $GL(V, E)$  as  $\varinjlim GL(V_n)$  and  $SL(V, E)$  as  $\varinjlim SL(V_n)$ . When we write  $GL(\infty; \mathbb{C})$  or  $SL(\infty; \mathbb{C})$  we must have in mind such an associated exhaustion of  $V$  by finite dimensional subspaces.

For the orthogonal and symplectic groups,  $V$  is endowed with a nondegenerate symmetric or antisymmetric bilinear form  $b$  that is related to  $E$  as follows: We can choose the increasing union  $E = \bigcup E_n$  so that the  $V_n = \text{Span}\{E_n\}$  are nondegenerate for  $b$ , and so that  $b(e_m, V_n) = 0$  for  $e_m \notin E_n$ . Thus  $O(V, E, b) = \varinjlim O(V_n, b|_{V_n})$  when  $b$  is symmetric, and  $Sp(V, E, b) = \varinjlim Sp(V_n, b|_{V_n})$  when  $b$  is antisymmetric. Again, when we write  $O(\infty; \mathbb{C})$ ,  $SO(\infty; \mathbb{C})$  or  $Sp(\infty; \mathbb{C})$  we must have in mind such an associated exhaustion of  $V$  by finite dimensional  $b$ -nonsingular subspaces.

## 2.2 Flags.

We now recall some basic definitions from [2]. A **chain** of subspaces in  $V$  is a set  $\mathcal{C}$  of distinct subspaces such that if  $F, F' \in \mathcal{C}$  then either  $F \subset F'$  or  $F' \subset F$ . We write  $\mathcal{C}'$  (resp.  $\mathcal{C}''$ ) for the subchain of all  $F \in \mathcal{C}$  with an immediate successor (resp. immediate predecessor). Also, we write  $\mathcal{C}^\dagger$  for the set of all pairs  $(F', F'')$  where  $F'' \in \mathcal{C}''$  is the immediate successor of  $F' \in \mathcal{C}'$ .

Let  $\mathcal{F}$  be a chain, and let  $\mathcal{F}'$  and  $\mathcal{F}''$  be defined as just above. Then  $\mathcal{F}$  is a **generalized flag** if  $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$  and  $V \setminus \{0\} = \bigcup_{(F', F'') \in \mathcal{F}^\dagger} (F'' \setminus F')$ . Note that  $0 \neq v \in V$  determines  $(F', F'') = (F'_v, F''_v) \in \mathcal{F}^\dagger$  such that  $v \in F'' \setminus F'$ . If  $\mathcal{F}$  is a generalized flag then each of  $\mathcal{F}'$  and  $\mathcal{F}''$  determines  $\mathcal{F}$ :

$$\text{if } (F', F'') \in \mathcal{F}^\dagger \text{ then } F' = \bigcup_{G'' \in \mathcal{F}'', G'' \subsetneq F''} G'' \text{ and } F'' = \bigcap_{G' \in \mathcal{F}', G' \supsetneq F'} G'.$$

A generalized flag  $\mathcal{F}$  is **maximal** if it is not properly contained in another generalized flag. This is equivalent to the condition that  $\dim F''_v / F'_v = 1$  for all  $0 \neq v \in V$ .

A generalized flag is a **flag** if, as a linearly ordered set, the proper subspaces of  $\mathcal{F}$  are isomorphic to a linearly ordered subset of  $\mathbb{Z}$ , so that we don't have to deal with limit ordinals.

In the orthogonal and symplectic cases, we say that a generalized flag  $\mathcal{F}$  in  $V$  is **isotropic** (relative to  $b$ ) if  $b(F, F) = 0$  for every  $F \in \mathcal{F}$ . This is equivalent to the notion in [7], where “isotropic” is defined to mean that  $\tau : F \mapsto F^\perp$  (relative to  $b$ ) is an order-reversing involution of  $\mathcal{F}$ , so that  $(F', F'') \in \mathcal{F}^\dagger$  if and only if  $((F'')^\perp, (F')^\perp) \in \mathcal{F}^\dagger$ . In effect, if  $\mathcal{F} = (F_\alpha)$  is isotropic in the sense of this paper then  $\mathcal{F} \cup \mathcal{F}^\perp := \mathcal{F} \cup \{F^\perp \mid F \in \mathcal{F}\}$  is isotropic in the sense of [7], and if  $\mathcal{J} = \{J_\beta\}$  is isotropic in the sense of [7] then  $\{J_\alpha \mid J_\alpha \subset J_\alpha^\perp\}$  is isotropic in the current sense.

A partial order  $\prec$  on a basis  $E$  of  $V$  is called **strict** if  $\beta \prec \alpha$  implies  $\beta \neq \alpha$ , and  $\beta \preceq \alpha$  means that either  $\beta \prec \alpha$  or  $\beta = \alpha$ . We emphasize that this is only a partial order, not a linear order, and there may be elements of the index set that are not comparable under  $\prec$ . In particular  $\prec$  need not be the same as any order with which  $E$  is presented. See Example 2.2.2 below.

**Definition 2.2.1.** A generalized flag  $\mathcal{F}$  is **compatible** with  $E$  if there exists a strict partial order  $\prec$  on  $E$  for which every pair  $(F', F'')$  is a pair  $(\text{Span}\{e_\beta \mid \beta \prec \alpha\}, \text{Span}\{e_\beta \mid \beta \preceq \alpha\})$  or a pair  $(0, \text{Span}\{e_\beta \mid \beta \preceq \alpha\})$ . If  $\mathcal{F}$  is isotropic in the sense that each  $F_\alpha$  is either isotropic or coisotropic, then in addition we require that  $E$  be isotropic.

A generalized flag  $\mathcal{F}$  is **weakly compatible** with  $E$  if it is compatible with a basis  $L$  of  $V$  where  $E \setminus (E \cap L)$  is finite.

A subspace  $F \subset V$  is **(weakly) compatible** with  $E$  if the generalized flag  $(0, F, V)$  is (weakly) compatible with  $E$ .

Generalized flags  $\mathcal{F}$  and  $\mathcal{G}$  are  **$E$ -commensurable** if they are both weakly compatible with  $E$  and there is a bijection  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  and a finite dimensional  $U \subset V$  such that each  $F \subset \varphi(F) + U$ ,  $\varphi(F) \subset F + U$ , and  $\dim(F \cap U) = \dim(\varphi(F) \cap U)$ .  $E$ -commensurability is an equivalence relation.  $\diamond$

**Example 2.2.2.** This is the example that we'll need to discuss bounded symmetric domains. Let  $\mathcal{F} = (0 \subset F \subset V)$ . We divide the index set  $A$  of the basis  $E$  as  $A = A_1 \cup A_2$  where  $A_1 = \{\alpha \mid e_\alpha \in F\}$ . Let  $\prec$  be any partial order on  $A$  such that (i)  $\alpha_1 \prec \alpha_2$  whenever  $\alpha_1 \in A_1$  and  $\alpha_2 \in A_2$  and (ii)  $A_i$  has a maximal element  $\gamma_i$  in the sense that  $\alpha \prec \gamma_i$  whenever  $\gamma_i \neq \alpha \in A_i$ . Then  $(0, F) = (0, \text{Span}\{e_\beta \mid \beta \preceq \gamma_1\})$  (by convention on pairs with  $F' = 0$ ) and  $(F, V) = (\text{Span}\{e_\beta \mid \beta \prec \gamma_2\}, \text{Span}\{e_\beta \mid \beta \preceq \gamma_2\})$ , so  $\mathcal{F}$  is compatible with  $E$ .

This example extends to generalized flags of the form  $(0 \subset F_1 \subset \dots \subset F_\ell \subset V)$ , with only the obvious changes.  $\diamond$

Fix a generalized flag  $\mathcal{F}$  compatible with  $E$ . If  $E$  is  $b$ -isotropic suppose that  $\mathcal{F}$  is isotropic. Then  $\mathcal{Z} = \mathcal{Z}_{\mathcal{F},E}$  denotes the **flag manifold**  $G/Q$  where  $Q$  is the parabolic  $\{g \in G \mid g(F) = F \text{ for all } F \in \mathcal{F}\}$ . If  $E$  is isotropic we'll write  $\mathcal{Z} = \mathcal{Z}_{\mathcal{F},b,E}$  for  $G/Q$  where  $Q = Q_{\mathcal{F}}$  is the stabilizer of  $\mathcal{F}$  in  $G$ . As noted in Section 2.3,  $\mathcal{Z}_{\mathcal{F},E}$  is a holomorphic direct limit of finite dimensional complex flag manifolds, so  $\mathcal{Z}_{\mathcal{F},E}$  has the structure of complex manifold.

Theorem 6.2 in [2] says

**Lemma 2.2.3.** *Let  $\mathcal{F}$  be a generalized flag that is weakly compatible with  $E$ . If  $G = SO(V, E, b)$  or  $G = Sp(V, E, b)$  suppose further that  $\mathcal{F}$  is isotropic. If  $g \in G$ , then  $g(\mathcal{F})$  is  $E$ -commensurable to  $\mathcal{F}$ . If  $\mathcal{F}$  and  $\mathcal{L}$  are  $E$ -commensurable then there is an element  $g \in G$  such that  $\mathcal{L} = g(\mathcal{F})$ .*

*Proof.* (Compare with Theorem 6.1 of [2].) If  $g \in G$  then  $V = U + W$  where  $g$  is the identity on  $W$ ,  $g(U) = U$ , and  $\dim U < \infty$ . If  $F_{\alpha} \in \mathcal{F}$  then  $g(F_{\alpha}) \subset F_{\alpha} + U$ . In particular  $g(\mathcal{F})$  is weakly compatible with  $E$ . This proves the first statement.

Let  $\mathcal{F}$  and  $\mathcal{L}$  be  $E$ -commensurable, and let  $U$  be a finite dimensional subspace of  $V$ , such that each  $F \subset \varphi(F) + U$ ,  $\varphi(F) \subset F + U$  and  $\dim(F \cap U) = \dim(\varphi(F) \cap U)$ . They are weakly compatible with  $E$  so they are compatible with bases  $X$  and  $Y$  such that  $E \setminus (E \cap X)$  and  $E \setminus (E \cap Y)$  are finite. Now  $E \setminus (E \cap (X \cup Y))$  is finite; let  $U$  denote its span and let  $W$  be the span of its complement in  $E$ . Let  $g \in G$  be the identity on  $W$ , and define  $g : U \rightarrow U$  by  $g(x_{\alpha}) = y_{\alpha}$  for  $\alpha$  an index of  $E \setminus (E \cap (X \cup Y))$ .  $\square$

## 2.3 Flag Manifolds.

Let  $\mathcal{F}$  be a generalized flag weakly compatible with  $E$ . If  $G = SO(V, E, b)$  or  $G = Sp(V, R, b)$ , suppose that  $\mathcal{F}$  is isotropic. In view of Lemma 2.2.3,

**Remark 2.3.1.** The **flag manifold**  $\mathcal{Z}_{\mathcal{F},E}$  consists of all generalized flags in  $V$  that are  $E$ -commensurable to  $\mathcal{F}$ .  $\diamond$

Lemma 2.2.3 says that  $\mathcal{Z}_{\mathcal{F},E}$  is a homogeneous space for the complex group  $G$ . Realize  $V = \varinjlim V_n$  according to an exhaustion  $E = \bigcup E_n$  by finite subsets. Denote  $\mathcal{F}_n = \mathcal{F} \cap V_n$ . In other words, if  $\mathcal{F} = \{F_{\alpha}\}_{\alpha \in A}$  then  $\mathcal{F}_n$  is  $\{F_{\alpha} \cap V_n\}_{\alpha \in A}$  with repetitions allowed. Now  $\mathcal{F}_n$  is a flag in  $V_n$  so we have the flag manifold  $\mathcal{Z}_{\mathcal{F}_n, E_n}$ . Note that the  $F_{\alpha} \cap V_n \hookrightarrow F_{\alpha} \cap V_m$ ,  $m \geq n$ , define maps  $\mathcal{Z}_{\mathcal{F}_n, E_n} \rightarrow \mathcal{Z}_{\mathcal{F}_m, E_m}$  and give us a direct system  $\{\mathcal{Z}_{\mathcal{F}_n, E_n}\}$  for which  $\mathcal{Z}_{\mathcal{F},E} = \varinjlim \{\mathcal{Z}_{\mathcal{F}_n, E_n}\}$ . Since the finite dimensional flag manifold  $\mathcal{Z}_{\mathcal{F}_n, E_n}$  has the natural structure of homogeneous projective variety under the action of  $G_n$ , and the  $\mathcal{Z}_{\mathcal{F}_n, E_n} \rightarrow \mathcal{Z}_{\mathcal{F}_m, E_m}$  are equivariant rational maps and equivariant for  $G_n \hookrightarrow G_m$ , the infinite dimensional flag manifold  $\mathcal{Z}_{\mathcal{F},E}$  is a  $G$ -homogeneous ind-variety. We emphasize the connection with the (finite dimensional)  $\mathcal{Z}_{\mathcal{F}_n, E_n}$  by viewing  $\mathcal{Z}_{\mathcal{F},E}$  as a complex ind-manifold referring to it simply as a complex flag manifold.

## 2.4 Real Forms of the Complex Groups.

Corresponding to the complex classical groups  $G$  mentioned above, we have their real forms as follows. Here note that a local isomorphism to one of the groups on the following list implies an isomorphism of Lie algebras, so the local isomorphism is compatible with the ind-structure specified as direct limit of finite dimensional Lie groups.

If  $G = SL(\infty; \mathbb{C})$ , then  $G_0$  is locally isomorphic to one of

- $SL(\infty; \mathbb{R}) = \lim_{n \rightarrow \infty} SL(n; \mathbb{R})$  the real special linear group,
- $SL(\infty; \mathbb{H}) = \lim_{n \rightarrow \infty} SL(n; \mathbb{H})$  the quaternion special linear group,
- $SU(p, \infty) = \lim_{n \rightarrow \infty} SU(p, n)$  the complex special unitary group of finite real rank  $p$ , and
- $SU(\infty, \infty) = \lim_{p, q \rightarrow \infty} SU(p, q)$  the complex special unitary group of infinite real rank.

If  $G = GL(\infty; \mathbb{C})$ , then  $G_0$  is locally isomorphic to one of

- $GL(\infty; \mathbb{R}) = \lim_{n \rightarrow \infty} GL(n; \mathbb{R})$  the real general linear group,
- $GL(\infty; \mathbb{H}) = \lim_{n \rightarrow \infty} GL(n; \mathbb{H}) = SL(\infty; \mathbb{H}) \times \mathbb{R}$  the quaternion general linear group,
- $U(p, \infty) = \lim_{n \rightarrow \infty} U(p, n)$  the complex unitary group algebra of finite real rank  $p$ , and
- $U(\infty, \infty) = \lim_{p, q \rightarrow \infty} U(p, q)$  the complex unitary group of infinite real rank.

If  $G = SO(\infty; \mathbb{C})$ , then  $G_0$  is locally isomorphic to one of

- $SO(p, \infty) = \lim_{n \rightarrow \infty} SO(p, n)$  the real orthogonal group of finite real rank  $p$ ,
- $SO(\infty, \infty) = \lim_{p, q \rightarrow \infty} SO(p, q)$  the real orthogonal group of infinite real rank, and
- Caveat: when we write  $SO(-)$  we mean the topological identity component of  $O(-)$ .*
- $SO^*(\infty) = \lim_{n \rightarrow \infty} (SO^*(2n) = \{g \in SL(n; \mathbb{H}) \mid g \text{ preserves } \kappa(x, y) := \sum \bar{x}^\ell i y^\ell = {}^t \bar{x} i y\})$ .

If  $G = Sp(\infty; \mathbb{C})$ , then  $G_0$  is locally isomorphic to one of

- $Sp(\infty; \mathbb{R}) = \lim_{n \rightarrow \infty} Sp(n; \mathbb{R})$  the real symplectic group,
- $Sp(p, \infty) = \lim_{n \rightarrow \infty} Sp(p, n)$  the quaternion unitary Lie algebra of finite real rank  $p$ , and
- $Sp(\infty, \infty) = \lim_{p, q \rightarrow \infty} Sp(p, q)$  the quaternion unitary Lie algebra of infinite real rank.

As usual we use Roman letters for the Lie groups and the corresponding lower case fraktur for their Lie algebras. In order to be precise about the real groups we must be careful about two notions: nondegeneracy of subspaces, and the role of  $V$  and  $E$  in complex conjugation  $\tau$  of  $\mathfrak{g}$  over  $\mathfrak{g}_0$  and  $G$  over  $G_0$ .

### 3 Basis and Exhaustion.

We run through the real groups of Section 2.4, defining some particular bases, flags and signatures relevant to our results on cycle spaces.

#### 3.1 $SU(\infty, q)$ , $q \leq \infty$ .

In this case  $V = \mathbb{C}^{\infty, q}$  with  $q \leq \infty$  and we start with an ordered basis

$$(3.1.1) \quad \begin{aligned} E &= \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots, e_q\} \text{ if } q < \infty, \\ E &= \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots\} \text{ if } q = \infty, \end{aligned}$$

where  $G_0$  is defined by the hermitian form

$$(3.1.2) \quad h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

The corresponding exhaustion  $V = \bigcup V_n$  realizes  $G_0$  as  $\lim_{k, \ell \rightarrow \infty} SU(k, \ell)$  or  $\lim_{k \rightarrow \infty} SU(k, q)$ .

$\mathcal{F}$  is a flag in  $V$  compatible with the ordered basis  $E$  of (3.1.1). *The partial order  $\prec$  for this compatibility is not necessarily the order of (3.1.1); it is a property of  $\mathcal{F}$  relative to  $E$  rather than a property of the ordering (3.1.1) of  $E$ .* To each flag  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$  we assign the signature sequence  $\{s_k = s_k(\mathcal{F}^{(1)}) := (\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}), \text{nul}_k(\mathcal{F}^{(1)}))\}$  where  $\text{pos}_k$  is the dimension of the maximal positive definite subspace of  $F_k^{(1)}$ ,  $\text{neg}_k$  is the dimension of the maximal negative definite subspace, and  $\text{nul}_k$  is the nullity. If  $\text{nul}_k = 0$  we write  $(\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}))$  for  $(\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}), 0)$ . If the  $F_k$  all are finite dimensional, then  $\text{pos}_k$ ,  $\text{neg}_k$  and  $\text{nul}_k$  all are finite. If  $q < \infty$ , i.e. if  $G_0$  has finite real rank  $q$ , then every  $\text{nul}_k \leq q$ . However, when one or more of the  $F_k$  is infinite dimensional the signature sequence is not always useful.

### 3.2 $SO(\infty, q), q \leq \infty$ .

Again  $V = \mathbb{C}^{\infty, q}$ . In terms of an ordered basis  $E' = \{e'_i\}$  as in (3.1.1),  $G_0$  is defined by a symmetric bilinear form  $b$  together with a hermitian form  $h$ , as follows:

$$(3.2.1) \quad \begin{aligned} b(e'_i, e'_j) &= +\delta_{i,j} = h(e'_i, e'_j) \text{ for } i < 0, \quad b(e'_i, e'_j) = -\delta_{i,j} = h(e'_i, e'_j) \text{ for } i > 0, \\ \text{the other } b(e'_k, e'_\ell) &= 0 = h(e'_k, e'_\ell). \end{aligned}$$

To see that (3.2.1) defines  $SO(\infty, q)$ , we note that  $SO(\infty, q)$  consists of all finitary real matrices (relative to the basis  $E'$ ) in the  $SO(\infty; \mathbb{C})$  defined by  $b$ , and also consists of all real matrices in the  $SU(\infty, q)$  defined by  $h$ . Write  $B$  and  $H$  for the matrices of  $b$  and  $h$ , so  $SO(\infty; \mathbb{C})$  is given by  $gB \cdot {}^t g = B$  and  $SU(\infty, q)$  is given by  $gH \cdot {}^t \bar{g} = H$ . Since  $B = H$ , now  $g \in SO(\infty; \mathbb{C}) \cap SU(\infty, q)$  implies  $g = \bar{g}$  so  $g \in SO(\infty, q)$ , and obviously  $g \in SO(\infty, q)$  implies  $g \in SO(\infty; \mathbb{C}) \cap SU(\infty, q)$ . For  $k, \ell \leq \infty$  we have verified

$$(3.2.2) \quad SO(k, \ell) = SO(k + \ell; \mathbb{C}) \cap SU(k, \ell), \quad SO(k + \ell; \mathbb{C}) \text{ defined by } b, \quad SU(k, \ell) \text{ defined by } h.$$

A  $b$ -isotropic flag in  $V$  cannot be compatible with  $E'$  because a subspace of  $V$  spanned by some of the  $e'_i$  neither contains nor is contained in its  $b$ -orthocomplement. So we define

$$(3.2.3) \quad \begin{aligned} E &= \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots, e_q\} \text{ if } q < \infty, \\ E &= \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots\} \text{ if } q = \infty, \end{aligned}$$

where

$$(3.2.4) \quad \begin{aligned} h(e_i, e_j) &= \delta_{i,j} \text{ for } i < 0, \quad h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0 \text{ and} \\ b(e_i, e_j) &= \delta_{i,j} \text{ for } i < -q, \quad b(e_i, e_j) = \delta_{0, i+j} \text{ for } i \geq -q. \end{aligned}$$

The transformation  $e'_i \mapsto e_i$  is not finitary when  $q = \infty$ , but nonetheless every  $g \in G$  is finitary relative to the basis  $E$ .  $\mathcal{F}$  is an  $E$ -commensurable isotropic flag in  $V$ . We use  $h$  for the signature sequence  $\{s_k = s_k(\mathcal{F}^{(1)}) := (\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}), \text{nul}_k(\mathcal{F}^{(1)}))\}$  for a flag  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$ , where  $\text{pos}_k$  is the dimension of the maximal  $h$ -positive definite subspace of  $F_k^{(1)}$ ,  $\text{neg}_k$  is the dimension of the maximal  $h$ -negative definite subspace, and  $\text{nul}_k$  is the  $h$ -nullity. As in Section 3.1 above, if  $\text{nul}_k = 0$  we write  $(\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}))$  for  $(\text{pos}_k(\mathcal{F}^{(1)}), \text{neg}_k(\mathcal{F}^{(1)}), 0)$ , and if the  $F_k$  all are finite dimensional, then  $\text{pos}_k$ ,  $\text{neg}_k$  and  $\text{nul}_k$  all are finite, and if  $q < \infty$  then every  $\text{nul}_k \leq q$ .

**Remark 3.2.5.** Orientation can be a consideration for  $SO(\infty, q)$ . Following [1, Theorem 2.8], the stabilizer of a  $b$ -isotropic flag  $\mathcal{F}$  determines all the subspaces in  $\mathcal{F}$  except when some there is an isotropic subspace  $L \in \mathcal{F}$  with  $\dim L^\perp / L = 2$ . In that case there are two maximal isotropic subspaces  $M_1$  and  $M_2$  of  $(V, b)$  that contain  $L$ , and there are three flags with the same stabilizer as  $\mathcal{F}$ , and of course  $\mathcal{F}$  is one of them. We list them with *ad hoc* designations. (i) (undecided orientation)  $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\}$  and neither of the  $M_i$  is contained in  $\mathcal{F}$ , (ii) (positive orientation)  $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\} \cup \{M_1\}$ , i.e.  $M_1 \in \mathcal{F}$ , and (iii) (negative orientation)  $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\} \cup \{M_2\}$ , i.e.  $M_2 \in \mathcal{F}$ . Signature does not distinguish these three flags, for example (ii) and (iii) have the same signatures for all  $q$ , and sometimes all three have the same signature with  $q = \infty$ .

### 3.3 $Sp(\infty, q), q \leq \infty$ .

Here  $V = \mathbb{C}^{\infty, 2q}$  and we use the basis (3.1.1) with  $q$  replaced by  $2q$ . Then  $G_0$  is defined by both an antisymmetric bilinear form  $b$  and an hermitian form  $h$ .

$$(3.3.1) \quad \begin{aligned} b(e_{2i-1}, e_{2i}) &= -1, \quad b(e_{2i}, e_{2i-1}) = +1, \quad \text{for } i > 0, \\ b(e_{2i+1}, e_{2i}) &= +1, \quad b(e_{2i}, e_{2i+1}) = -1 \text{ for } i < 0, \quad \text{all other } b(e_a, e_b) = 0; \\ h(e_i, e_j) &= \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0, \quad \text{all other } h(e_a, e_b) = 0. \end{aligned}$$

To see this we need the analog of (3.2.2), and for that we need to find the quaternion algebra that realizes  $\mathbb{C}^{2p,2r}$  as  $\mathbb{H}^{p,r}$ .

**Lemma 3.3.2.**  $Sp(p, r) = Sp(p + r; \mathbb{C}) \cap SU(2p, 2r)$  where  $SO(p + r; \mathbb{C})$  is defined by  $b$  as in (3.3.1) and  $SU(2p, 2r)$  is defined by  $h$  as in (3.3.1).

*Proof.* We work in matrices relative to the portion  $E = \{e_{-2p}, \dots, e_{2r}\}$  of (3.1.1). Then  $b$  has matrix  $B = \text{diag}\{J, \dots, J\}$  where  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $h$  has matrix  $H = \begin{pmatrix} I_{2p} & 0 \\ 0 & -I_{2r} \end{pmatrix}$ . So  $Sp(p + r; \mathbb{C})$  is given by  $gB^t g = B$  and  $SU(2p, 2r)$  is given by  $gH^t \bar{g} = H$ . Define  $\mathbb{R}$ -linear transformations of  $V$  by

$$\mathcal{I}(v) = \sqrt{-1}v \text{ and } \mathcal{J}(v) = \sqrt{-1}BH\bar{v} \text{ for } v \in V.$$

Compute

$$\mathcal{I}^2 = -I, \mathcal{J}^2 = -I \text{ and } \mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I} = 0$$

so  $\mathcal{I}$  and  $\mathcal{J}$  generate a quaternion algebra; call it  $\mathbb{H}$ . If  $g \in Sp(p + r; \mathbb{C}) \cap SU(2p, 2r)$ , so  ${}^t g = B^{-1}g^{-1}B$  and  $\bar{g} = H^{-1}g^{-1}H$ , then  $B^{-1} = -B$ ,  $H^{-1} = H$ , and we compute

$$\begin{aligned} \mathcal{J}g\mathcal{J}^{-1}v &= (\sqrt{-1}BH)(\bar{g})(-\sqrt{-1}HB\bar{v}) = -BH\bar{g}HBv \\ &= -BH \cdot H^t g^{-1} H \cdot HBv = B^t g^{-1} Bv = B \cdot BgB^{-1} \cdot Bv = gv \end{aligned}$$

for  $v \in V$ . Thus  $\mathcal{J}$  commutes with every  $g \in Sp(p + r; \mathbb{C}) \cap SU(2p, 2r)$ , in other words every  $g \in Sp(p + r; \mathbb{C}) \cap SU(2p, 2r)$  is  $\mathbb{H}$ -linear. That shows  $Sp(p + r; \mathbb{C}) \cap SU(2p, 2r) \subset Sp(p, r)$ .

On the other hand,  $\sigma : g \mapsto \mathcal{J}g\mathcal{J}^{-1}$  is an involutive automorphism on the underlying real structure of  $Sp(p + r; \mathbb{C})$ . The latter is simply connected, so its fixed point set is connected. But  $\sigma$  fixes every element of  $Sp(p, r)$ , which is maximal among the connected subgroups of  $Sp(p + r; \mathbb{C})$ . So now  $Sp(p, r) \subset Sp(p + r; \mathbb{C}) \cap SU(2p, 2r)$ . That completes the proof.  $\square$

Now take the limit on  $p$ , or on  $p$  and  $r$ , to see how  $G_0$  is defined by the two forms  $b$  and  $h$  of (3.3.1). Let  $\mathcal{F}$  be a  $b$ -isotropic flag in  $V$  compatible with the basis  $E$  of (3.1.1). For that, note that  $\text{Span}\{e_i \mid i \text{ even}\}$  and  $\text{Span}\{e_i \mid i \text{ odd}\}$  are  $b$ -isotropic subspaces. As in the previous cases one can discuss signature for flags  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$ .

### 3.4 $SO^*(\infty)$ .

This case is similar to the case of  $SO(\infty, \infty)$ , except that we use a different bilinear form  $b$ . The basis is

$$(3.4.1) \quad E = \{\dots e_{-3}, e_{-2}, e_{-1}, e_1, e_2, e_3, \dots\} = \bigcup E_n \text{ where } E_n = \{e_{-n}, \dots, e_n\}.$$

$G_0$  is defined by the symmetric bilinear form  $b$  and the hermitian form  $h$ :

$$(3.4.2) \quad b(e_i, e_j) = \delta_{i+j, 0}, \quad h(e_i, e_j) = \delta_{i, j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i, j} \text{ for } i > 0.$$

Thus  $V = \bigcup V_n$  where  $V_n = \text{Span}\{E_n\}$  and  $G_0 = SO^*(\infty) = \varinjlim SO^*(2n)$  where  $SO^*(2n)$  is the subgroup of  $SL(2n; \mathbb{C})$  defined by the forms  $b$  and  $h$  of (3.4.2). To check this, note that that subgroup of  $SL(2n; \mathbb{C})$  itself has maximal compact subgroup isomorphic to  $U(n)$ .

### 3.5 $Sp(\infty; \mathbb{R})$ .

This case is similar to the case of  $SO^*(\infty)$ , except that the bilinear form  $b$  is antisymmetric. We use the same basis (3.4.1), with bilinear form  $b$  and hermitian form  $h$ :

$$(3.5.1) \quad \begin{aligned} b(e_i, e_j) &= \delta_{i+j,0} \text{ for } i < 0 \text{ and } b(e_i, e_j) = -\delta_{i+j,0} \text{ for } i > 0; \\ h(e_i, e_j) &= \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0. \end{aligned}$$

Here  $G = Sp(\infty; \mathbb{C})$  is defined by  $b$ . One can view  $G_0$  as the real (relative to  $\text{Span}_{\mathbb{R}}(E)$ ) elements of  $G$ , but for our purposes it is better to view it as  $G \cap SU(\infty, \infty)$  where  $U(\infty, \infty)$  is defined by the hermitian form  $h$ . For that, it suffices to check that  $Sp(n; \mathbb{R}) = Sp(n; \mathbb{C}) \cap U(n, n)$ , and to check that it suffices to note that  $Sp(n; \mathbb{C}) \cap SU(n, n)$  contains a  $U(n)$  in the form  $\begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix}$ .

As in the previous cases one can discuss signature for flags  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$ .

### 3.6 $SL(\infty; \mathbb{R})$ and $SL(\infty; \mathbb{H})$ .

Fix a real form  $V_0$  of  $V$  and an ordered basis  $E = \{e_1, e_2, e_3, \dots\}$  of  $V_0$ . Then  $SL(\infty; \mathbb{R})$  is defined by complex conjugation  $\tau : v \mapsto \bar{v}$  of  $V$  over a real form  $V_0$ , while  $SL(\infty; \mathbb{H})$  is defined by a conjugate linear map  $\tau : v \mapsto \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \bar{v}$  on  $V$ . In terms of  $E$ ,

Case  $\mathbb{F} = \mathbb{R}$ : each  $\tau e_a = e_a$ , so  $E$  is an  $\mathbb{R}$ -basis of a real form  $V_0$  of  $V$

Case  $\mathbb{F} = \mathbb{H}$ :  $\tau e_{2a-1} = -e_{2a}$  and  $\tau e_{2a} = e_{2a-1}$ , so each  $\{e_{2a-1}, ie_{2a-1}, e_{2a}, ie_{2a}\}$  is an  $\mathbb{R}$ -basis of an  $\mathbb{H}$ -subspace of  $V$

In the finite dimensional case the signature for a generalized flag  $\mathcal{F}^{(1)}$  is  $\{s_{i,j} = s_{i,j}(\mathcal{F}^{(1)})\}$  where  $s_{i,j}(\mathcal{F}^{(1)})$  is the dimension of the maximal complex subspace of  $F_i^{(1)} \cap \tau F_j^{(1)}$  ([4] and [5]). In the infinite dimensional cases we will have to be more precise [7, §5].

### 3.7 Nondegeneracy and Open Orbits.

Fix a basis  $E$  of  $V$  as in Sections 3.1 through 3.6, and a flag  $\mathcal{F}$  in  $V$  that is compatible with  $E$ . Except in the cases of  $SL(\infty; \mathbb{R})$  and  $SL(\infty; \mathbb{H})$ , we use signatures of generalized flags to distinguish real group orbits on  $\mathcal{Z}_{\mathcal{F}, E}$ , as follows.

**Definition 3.7.1.** Let  $G_0$  be defined by a nondegenerate bilinear form  $b$  or an hermitian form  $h$  or both. Then we say that a flag  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$  is **nondegenerate** if (i) for  $SU(\infty, q)$ ,  $SO(\infty, q)$  or  $Sp(\infty, q)$  each  $F_\alpha^{(1)}$  is  $h$ - (or  $b$ -) nondegenerate; and (ii) for  $Sp(n; \mathbb{R})$  or  $SO^*(\infty)$  each  $F_\alpha^{(1)}$  is  $h$ -nondegenerate.  $\diamond$

**Theorem 3.7.2.** Let  $G_0$  be  $SU(\infty, q)$ ,  $SO(\infty, q)$ ,  $Sp(\infty, q)$ ,  $SO^*(\infty)$  or  $Sp(\infty; \mathbb{R})$  and consider a flag  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$ . Then  $G_0(\mathcal{F}^{(1)})$  is open in  $\mathcal{Z}_{\mathcal{F}, e}$  if and only if  $\mathcal{F}^{(1)}$  is nondegenerate.

*Proof.* An orbit  $G_0(\mathcal{F}^{(1)})$  in  $\mathcal{Z}_{\mathcal{F}, e}$  is open just when one stays inside the orbit under any sufficiently small perturbation of a finite number of the  $F^{(1)}$  in  $\mathcal{F}^{(1)}$ . Using the direct limit topology on  $\mathcal{Z}_{\mathcal{F}, e}$  and the finite dimensional analog ([12], [3]), the assertion follows. This is the same argument as that of the first part of [7, Proposition 5.1].  $\square$

**Corollary 3.7.3.** There are open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F}, E}$  if and only if  $\mathcal{Z}_{\mathcal{F}, E}$  contains a nondegenerate flag. In particular, if  $G_0$  is  $SU(\infty, q)$ ,  $SO(\infty, q)$  or  $Sp(\infty, q)$  with  $q < \infty$  then there are open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F}, E}$ .

The matter is subtler for the special and general linear groups, where we don't have  $b$ - or  $h$ -nondegeneracy for subspaces of  $V$ , and where if  $\dim V = \infty$  then the  $\dim(F_i^{(1)} \cap \tau F_j^{(1)})$  do not suffice. Instead we use [7, Definition 5.1] as follows.



**Definition 3.7.4.** Let  $G_0$  be  $SL(\infty; \mathbb{F})$ ,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{H}$ . Then  $G_0(\mathcal{F}^{(1)}) \subset \mathcal{Z}_{\mathcal{F}, E}$  is **nondegenerate** if  $F_i^{(1)} \cap \tau F_j^{(1)}$  fails to properly contain  $F_i^{(2)} \cap \tau F_j^{(2)}$ , whenever  $\mathcal{F}^{(2)} \in \mathcal{Z}_{\mathcal{F}, E}$  and  $F_i^{(1)}, F_j^{(1)} \in \mathcal{F}^{(1)}$ .  $\diamond$

The first consequence of this definition is

**Theorem 3.7.5.** ([7, Proposition 5.3]) *Let  $G_0$  be  $SL(\infty; \mathbb{R})$  or  $SL(\infty; \mathbb{H})$ , and consider a flag  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$ . Then the orbit  $G_0(\mathcal{F}^{(1)})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$  if and only if  $\mathcal{F}^{(1)}$  is nondegenerate. In particular, if each  $F_n^{(1)} \cap \tau F_n^{(1)} = 0$  then  $G_0(\mathcal{F}^{(1)})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ .*

If  $n$  is odd, or if  $n = 2m$  and  $\dim F \neq m$  for all  $F \in \mathcal{F} \cap \mathbb{C}^n$ , then  $G_{n,0} = SL(n; \mathbb{R})$  has only one open orbit on a flag manifold  $G_n/Q_n$  in  $\mathbb{C}^n$ ; if  $n = 2m$  even, and some  $F \in \mathcal{F} \cap \mathbb{C}^n$  has dimension  $m$ , then there is an orientation question and  $G_{n,0} = SL(n; \mathbb{R})$  has two open orbits on  $G_n/Q_n$ . See [4, Corollary 2.3]. Further  $G_{n,0} = SL(n; \mathbb{H})$  has a unique open orbit on a flag manifold  $G_n/Q_n$  in  $\mathbb{C}^{2n}$ . See [5, Proposition 3.14]. This extends to infinite dimensions as follows.

**Corollary 3.7.6.** *Let  $G_0$  be  $SL(\infty; \mathbb{R})$  or  $SL(\infty; \mathbb{H})$ . Then there is an open  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$  if and only if  $\mathcal{Z}_{\mathcal{F}, E}$  contains a nondegenerate flag, and in that case there is exactly one open  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$ .*

*Proof.* The first assertion is immediate from Theorem 3.7.5. For the second, let  $\mathcal{O}_1 = G_0(\mathcal{F}^{(1)})$  and  $\mathcal{O}_2 = G_0(\mathcal{F}^{(2)})$  be open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F}, E}$ . Then

$$\begin{aligned} \mathcal{Z}_{\mathcal{F}, E} &= \varinjlim \mathcal{Z}_{\mathcal{F}_n, E_n} \text{ where, for } n \text{ in a cofinal subset } \mathbb{S} \subset \mathbb{Z}^+, \\ E &\text{ is an increasing union of finite subsets } E_n, \\ V_n &= \text{Span}\{E_n\} \text{ and } \mathcal{F}_n = \mathcal{F} \cap V_n := (F_k \cap V_n), \\ \mathcal{Z}_{\mathcal{F}_n, E_n} &= G_n/Q_n \text{ flag manifold in } V_n \text{ with } Q_n \text{ parabolic in } G_n, \text{ and} \\ \mathcal{O}_k \cap \mathcal{Z}_{\mathcal{F}_n, E_n} &\text{ is an open } G_{n,0}\text{-orbit on } \mathcal{Z}_{\mathcal{F}_n, E_n} \text{ for } k = 1, 2. \end{aligned}$$

In the  $SL(\infty; \mathbb{R})$  case we modify  $\mathbb{S}$ . If  $n \in \mathbb{S}$  is even and  $n+1 \notin \mathbb{S}$  we replace  $n$  by  $n+1$ . If  $n \in \mathbb{S}$  is even and  $n+1 \in \mathbb{S}$  we delete  $n$ . Thus we may assume that every element of  $\mathbb{S}$  is odd. In the  $SL(\infty; \mathbb{H})$  case we do not modify  $\mathbb{S}$ . Thus, in both cases, if  $n \in \mathbb{S}$  then  $G_{n,0}$  has a unique open orbit on  $\mathcal{Z}_{\mathcal{F}_n, E_n}$ , so  $(\mathcal{O}_1 \cap \mathcal{Z}_{\mathcal{F}_n, E_n}) = (\mathcal{O}_2 \cap \mathcal{Z}_{\mathcal{F}_n, E_n})$ . Thus  $\mathcal{O}_1$  meets  $\mathcal{O}_2$ , so  $\mathcal{O}_1 = \mathcal{O}_2$ .  $\square$

Combining the argument of the proof of Corollary 3.7.6 with the uniqueness of closed orbits in the finite dimensional case [12], we have the related result

**Proposition 3.7.7.** (Compare [7, Proposition 5.6].) *Let  $G_0$  be  $SL(\infty; \mathbb{R})$  or  $SL(\infty; \mathbb{H})$ . Then there is closed  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$  if and only if each  $\tau F_i = F_i$ , and in that case there is exactly one closed  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$ .*

## 4 Complex Bounded Symmetric Domains

The bounded symmetric domains are important cases of the orbits considered in Section 3. In finite dimensions they play a pivotal role in complex analysis, moduli theory, cycle space theory, automorphic function theory, and both riemannian and complex differential geometry. In this section we extend parts of the finite dimensional bounded domain theory to our infinite dimensional setting, following the lines of the classical examples in [13].

In the classical theory one has the bounded symmetric domain  $D_0 = G_0(z_0)$ , its compact dual hermitian symmetric space  $\mathcal{Z}$ , the Borel embedding  $D_0 \hookrightarrow \mathcal{Z}$ , and the Harish-Chandra embedding  $\xi^{-1}|_{D_0} : D_0 \hookrightarrow \mathfrak{m}^+$ . In the Harish-Chandra embedding,  $\mathfrak{m}^+ \subset \mathfrak{g}$  is a commutative

subalgebra that represents the holomorphic tangent space and  $\xi : \mathfrak{m}^+ \rightarrow \mathcal{Z}$  by  $\xi(X) = \exp(X)z_0$ . A maximal set of strongly orthogonal noncompact positive roots  $\{\alpha_1, \dots, \alpha_r\}$ ,  $r = \text{rank } D_0$ , defines a set  $\{c_1, \dots, c_r\}$  of partial Cayley transforms, and the  $G_0$ -orbits on  $\mathcal{Z}$  are exactly the  $G_0(c_1 \dots c_k c_{k+1}^2 \dots c_{k+\ell}^2 z_0)$  where  $k, \ell \geq 0$  and  $k + \ell \leq r$ . The open orbits are the  $G_0(c_1^2 \dots c_\ell^2 z_0)$ , i.e. the ones with  $k = 0$ , and  $G_0(c_1 \dots c_r z_0)$  is the Bergman–Shilov boundary of  $D_0$ . See [12]. It is not so difficult to verify that this theory goes through *mutatis mutandis* for the infinite dimensional bounded symmetric domains as well, with the one restriction that  $k + \ell < \infty$ .

There are only four classes of (finitary) infinite dimensional complex bounded symmetric domains: the  $SU(\infty, q)/S(U(\infty) \times U(q))$  with  $q \leq \infty$ , the  $Sp(\infty; \mathbb{R})/U(\infty)$ , the  $SO^*(\infty)/U(\infty)$ , and the  $SO(\infty, 2)/[SO(\infty) \times SO(2)]$ . Their respective symmetric space ranks are  $q, \infty, \infty$  and  $2$ . In the all four cases it is easier to use some linear algebra, as in the examples worked out in [13], than to stick to the general theory. But of course we indicate the connection. The fourth case, however, where  $\mathcal{Z}$  is a quadric in an infinite dimensional complex projective space, is not as straightforward as the others. Now we run through the cases.

#### 4.1 The Complex Bounded Symmetric Domain for $SU(\infty, q)$ .

We study the bounded symmetric domain  $D_0$  associated to  $G_0 = SU(\infty, q)$ ,  $q \leq \infty$ . Start with

$$\begin{aligned} E &= \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, \dots, e_q\} \text{ for } q < \infty \\ E &= \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\} \text{ for } q = \infty \end{aligned}$$

where the hermitian form  $h$  is given by

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0, \quad h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

Let  $F = \text{Span}\{e_i \mid i > 0\}$ . As in Example 2.2.2,  $\mathcal{F} = (0, F, V)$  is compatible with  $E$ . Also,  $G_0(\mathcal{F})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ . The bounded symmetric domain is  $D_0 = G_0(\mathcal{F}) \in \mathcal{Z}_{\mathcal{F}, E}$ . Note that  $\mathcal{Z}_{\mathcal{F}, E}$  is a complex Grassmann manifold and the domain is

$$(4.1.1) \quad D_0 = \{\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in \mathcal{Z}_{\mathcal{F}, E} \mid F^{(1)} \text{ is a maximal negative definite subspace of } V\}.$$

We go on to see why it is a bounded symmetric domain.

We will use the  $h$ -orthogonal decomposition  $V = V_+ \oplus V_-$  where  $V_+ = \text{Span}\{e_i \mid i < 0\}$  and  $V_- = \text{Span}\{e_i \mid i > 0\}$  and the orthogonal projections  $\pi_{\pm} : V \rightarrow V_{\pm}$ . The kernel of  $\pi_-$  is  $h$ -positive definite so it has zero intersection with  $F^{(1)}$  for any  $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in D_0$ . Thus  $\pi_- : F^{(1)} \cong V_-$  is injective. Since  $F^{(1)}$  is a maximal negative definite subspace  $\pi_- : F^{(1)} \cong V_-$  is surjective as well. Now we have a well defined linear map

$$(4.1.2) \quad Z_{F^{(1)}} : V_- \rightarrow V_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}.$$

Since  $\mathcal{F}^{(1)}$  is weakly compatible with  $E$ , the matrix of  $F^{(1)}$  relative to  $E$  has only finitely many nonzero entries. In other words  $Z_{F^{(1)}}$  is finitary. Using  $\pi_- : F^{(1)} \cong V_-$  and the basis  $\{e_i \mid i > 0\}$  of  $F = V_-$  we have a basis  $\{e_i''\}$  of  $F^{(1)}$  defined by  $\pi_-(e_i'') = e_i$ . Write  $e_i'' = e_i + \sum_{j < 0} z_{j,i} e_j$ ; then  $(z_{j,i})$  is the matrix of  $Z_{F^{(1)}}$ . The fact that  $F^{(1)}$  is  $h$ -negative definite, in other words  $(h(e_i'', e_\ell'')) \ll 0$ , translates to the matrix condition  $I - (z_{j,i})^*(z_{j,i}) \gg 0$ , equivalently the operator condition  $I - Z_{F^{(1)}}^* Z_{F^{(1)}} \gg 0$ .

Conversely if  $Z : V_- \rightarrow V_+$  is finitary and satisfies  $I - Z^* Z \gg 0$ , then the column span of its matrix relative to  $E$  is a maximal negative definite subspace  $F^{(1)}$ , and  $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in D_0$ .

The block form matrices of elements of  $G_0$  act by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} Z \\ I \end{pmatrix} \rightarrow \begin{pmatrix} AZ+B \\ CZ+D \end{pmatrix}$ , which has the same column span as  $\begin{pmatrix} (AZ+B)(CZ+D)^{-1} \\ I \end{pmatrix}$ . So  $G_0$  acts by linear fractional transformations,  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \rightarrow (AZ+B)(CZ+D)^{-1}$ . Now we summarize.

**Proposition 4.1.3.**  $D_0$  is realized as the bounded domain consisting of all finitary  $Z : V_- \rightarrow V_+$  such that  $I - Z^*Z \gg 0$ . In that realization the action of  $G_0$  is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \rightarrow (AZ+B)(CZ+D)^{-1}$ .

### Orbits

There are  $q+1$  open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F},E}$ :

$$D_k = G_0(0, F_{(k)}, V) \text{ where } F_{(k)} = \text{Span}\{e_{-k}, \dots, e_{-1}; e_{k+1}, \dots, e_q\} \text{ if } q < \infty, \\ F_{(k)} = \text{Span}\{e_{-k}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ if } q = \infty,$$

If  $q = 1$  then  $D_1$  and  $D_0$  are the upper and lower “hemispheres” in an infinite version of the Riemann sphere; they are related by the square of a Cayley transform. If  $q > 1$  then  $D_0$  is the only convex  $D_k$ , but the others are reached by squares of partial Cayley transforms applied to  $F = F_0$  as in [13], [8] and [11].

In this bounded symmetric domain setting, the  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F},E}$  of signature  $(a, b, c) = (\text{pos}, \text{neg}, \text{nul})$  have  $a$  and  $c$  finite and  $\leq q$  because each  $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in \mathcal{Z}_{\mathcal{F},E}$  is weakly compatible with  $E$ . We denote those orbits by

$$(4.1.4) \quad \begin{aligned} D_{a,b,c} &= G_0(0, (F_+ + F_- + F_0), V) \text{ where} \\ F_0 &= \text{Span}\{e_{-c} + e_c, \dots, e_{-1} + e_1\} \text{ (null)} \\ F_+ &= \text{Span}\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (positive)} \\ F_- &= \text{Span}\{e_{c+1}, \dots, e_{c+b}\} \text{ if } q < \infty, \text{Span}\{e_{c+1}, e_{c+2}, \dots\} \text{ if } q = \infty \text{ (negative)}. \end{aligned}$$

The open orbits are the  $D_a = D_{a,b,0}$ ,  $a < \infty$  and  $a+b = q$ . In other words, they are the ones for  $c = 0$ . If  $q < \infty$  there is a unique closed orbit,  $D_{0,0,q}$ , consisting of the  $\mathcal{F}^{(1)} = \{F^{(1)}\} \in \mathcal{Z}_{\mathcal{F},E}$  for which  $F^{(1)}$  is null. It is in the closure of every orbit. If  $q = \infty$  there is no closed orbit.

One goes from the initial orbit  $D_{0,q,0} = G_0(\mathcal{F})$  to any  $D_{a,b,c}$  by applying a product of partial Cayley transforms to  $F$ . Specifically,  $D_{a,b,c} = G_0(c_1 \dots c_c c_{c+1}^2 \dots c_{a+c}^2 \mathcal{F})$  where the partial Cayley transforms  $c_k$  (corresponding to  $0 \rightarrow 1 \rightarrow \infty \rightarrow -1 \rightarrow 0$  in one variable) are given by

$$(4.1.5) \quad c_k(e_{-k}) = \frac{1}{\sqrt{2}}(e_{-k} - e_k), \quad c_k(e_k) = \frac{1}{\sqrt{2}}(e_{-k} + e_k), \quad c_k(e_j) = e_j \text{ for } j \neq \pm k.$$

Here  $1 \leq k \leq q$  when  $q < \infty$  and  $1 \leq k < \infty$  when  $q = \infty$ . In particular one reaches the boundary (of  $D_0 = D_{0,q,0}$ ) orbits by a product without repetition of  $\leq q$  partial Cayley transforms, and if  $q < \infty$  the closed orbit is  $D_{0,0,q} = G_0(c_1 \dots c_q \mathcal{F})$ . If  $q < \infty$  the closed orbit is the Bergman-Shilov boundary of  $D_0$ .

## 4.2 The Complex Bounded Symmetric Domain for $Sp(\infty; \mathbb{R})$ .

Now let  $G_0 = Sp(\infty; \mathbb{R})$ . It is defined relative to the basis  $E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\}$  by the hermitian form  $h$  and the antisymmetric bilinear form  $b$ ,

$$\begin{aligned} h(e_i, e_j) &= \delta_{i,j} \text{ for } i < 0, \quad h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0 \text{ and} \\ b(e_i, e_j) &= \delta_{i+j,0} \text{ for } i < 0, \quad b(e_i, e_j) = -\delta_{i+j,0} \text{ for } i > 0. \end{aligned}$$

The domain  $D_0$  consists of the maximal  $h$ -negative definite  $b$ -isotropic subspaces of  $V$  in  $\mathcal{Z}_{\mathcal{F},E}$ . In other words, let  $F = \text{Span}\{e_i \mid i > 0\}$ . Evidently  $\mathcal{F} = (0, F, V)$  is compatible with  $E$  and  $G_0(\mathcal{F})$  is open in  $\mathcal{Z}_{\mathcal{F},E}$ . The bounded symmetric domain is

$$D_0 := G_0(\mathcal{F}) \subset \mathcal{Z}_{\mathcal{F},E}.$$

Note that  $\mathcal{Z}_{\mathcal{F},E}$  is contained in the complex Grassmann manifold of Section 4.1 for  $q = \infty$ .

In Lie group terms,  $D_0 \cong Sp(\infty; \mathbb{R})/U(\infty)$  where  $U(\infty)$  is the stabilizer of  $F$ . Let both  $\mathcal{F}$  and  $\mathcal{F}_{(0)}$  denote the flag  $(0, F, V)$ , so  $D_0 = G_0(\mathcal{F}_{(0)})$ . The open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F}, E}$  are the  $D_k = G_0(\mathcal{F}_{(k)})$  where

$$F_{(k)} = \text{Span}\{e_{-k}, e_{-k+1}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ and } \mathcal{F}_{(k)} = (0, F_{(k)}, V)$$

for integers  $k \geq 0$ . Note that  $D_k \cong Sp(\infty; \mathbb{R})/U(k, \infty - k)$  where the  $\infty - k$  refers to the action on  $\text{Span}\{e_{k+1}, e_{k+2}, \dots\}$ . Also,  $F_{(k)}^\perp = F_{(k)}$  relative to  $b$ , so  $\mathcal{F}_{(k)}^\perp = \mathcal{F}_{(k)}$ .

The corresponding  $D_\infty$  is the  $h$ -orthocomplement of  $D_0$ , orbit of  $(0, F_{(\infty)}, V)$  where  $F_{(\infty)} := \text{Span}\{e_i \mid i < 0\}$ .

As in the  $SU$  setting, the partial Cayley transforms  $c_j$  are given by (4.1.5) and one passes from  $D_0$  to  $D_k$  by  $F_{(k)} = c_1^2 c_2^2 \dots c_k^2 F_{(0)}$ . Compare [13]. Similarly, as in [11], the boundary of  $D_0$  is the union of the orbits  $G_0(c_1 c_2 \dots c_\ell F_{(0)})$ , but here there is no closed  $G_0$ -orbit on  $\mathcal{F}_{(k)}$ ,  $k < \infty$ , and thus no Bergman–Shilov boundary of  $D_0$ .

The calculations for  $D_0$  to be a bounded symmetric domain are essentially the same as those in Section 4.1. The result is

**Proposition 4.2.1.**  *$D_0$  is realized as the bounded domain consisting of all finitary  $Z : V_- \rightarrow V_+$  such that the matrix of  $Z$  is symmetric and  $I - Z^* Z \gg 0$ . In that realization the action of  $G_0$  is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \rightarrow (AZ + B)(CZ + D)^{-1}$ .*

**Corollary 4.2.2.** *The bounded symmetric domain for  $Sp(\infty; \mathbb{R})$  is a totally geodesic submanifold of the bounded symmetric domain for  $SU(\infty, \infty)$ .*

As in Section 4.1 for  $SU(\infty, \infty)$ , the  $G_0$ -orbit of signature  $(pos, neg, nul) = (a, b, c)$ ,  $a$  and  $c$  finite, is

$$\begin{aligned} D_{a,b,c} &= G_0(0, (F_+ + F_- + F_0), V) \text{ where} \\ F_0 &= \text{Span}\{(e_{-c} + e_c), \dots, (e_{-1} + e_1)\} \text{ (} h\text{-null),} \\ F_+ &= \text{Span}\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (} h\text{-positive definite),} \\ F_- &= \text{Span}\{e_{c+1}, e_{c+2}, \dots\} \text{ (} h\text{-negative definite),} \end{aligned}$$

and every  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$  is one of those  $D_{a,b,c}$ . We always have  $b = \dim F_- = \infty$ . The open orbits are the case  $c = 0$  mentioned above:  $D_k = D_{k,b,0}$ .

### 4.3 The Complex Bounded Symmetric Domain for $SO^*(\infty)$ .

Next, we let  $G_0 = SO^*(\infty)$ . It is defined relative to the basis  $E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\}$  by the hermitian form  $h$  and the symmetric bilinear form  $b$ ,

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0, \quad h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0, \quad \text{and } b(e_i, e_j) = \delta_{i+j,0} \text{ for all } i, j.$$

The domain  $D_0$  consists of the maximal  $h$ -negative definite  $b$ -isotropic subspaces of  $V$  that are weakly compatible with  $E$ . In other words, let  $F = \text{Span}\{e_i \mid i > 0\}$ . Evidently  $\mathcal{F} = (0, F, V)$  is compatible with  $E$  and  $G_0(\mathcal{F})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ . The bounded symmetric domain is

$$D_0 := G_0(\mathcal{F}) \subset \mathcal{Z}_{\mathcal{F}, E}.$$

Again,  $\mathcal{Z}_{\mathcal{F}, E}$  is contained in the complex Grassmann manifold of Section 4.1 for  $q = \infty$ .

In Lie group terms,  $D_0 \cong SO^*(\infty)/U(\infty)$  where  $U(\infty)$  is the stabilizer of  $F$ . Let both  $\mathcal{F}$  and  $\mathcal{F}_{(0)}$  denote the flag  $(0, F, V)$ , so  $D_0 = G_0(\mathcal{F}_{(0)})$ . The open  $G_0$ -orbits on  $\mathcal{Z}_{\mathcal{F}, E}$  are the  $D_k = G_0(\mathcal{F}_{(k)})$  where

$$F_{(k)} = \text{Span}\{e_{-k}, e_{-k+1}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ and } \mathcal{F}_{(k)} = (0, F_{(k)}, V)$$

for integers  $k \geq 0$ . Note that  $D_k \cong SO^*(\infty)/U(k, \infty - k)$  where the  $\infty - k$  refers to the action on  $\text{Span}\{e_{k+1}, e_{k+2}, \dots\}$ . Also,  $F_{(k)}^\perp = F_{(k)}$  relative to  $b$ , so  $\mathcal{F}_{(k)}^\perp = \mathcal{F}_{(k)}$ .

The corresponding  $D_\infty := G_0(\mathcal{F}_\infty)$  where  $F_\infty$  is the  $h$ -orthocomplement  $\text{Span}\{e_i \mid i < 0\}$  of  $F_0$ .

Here the partial Cayley transforms are not given by (4.1.5), but rather by

$$(4.3.1) \quad \begin{aligned} c_k(e_{-2k}) &= \frac{1}{\sqrt{2}}(e_{-2k} - e_{2k}), & c_k(e_{-2k+1}) &= \frac{1}{\sqrt{2}}(e_{-2k+1} + e_{2k-1}), \\ c_k(e_{2k-1}) &= \frac{1}{\sqrt{2}}(-e_{-2k+1} + e_{2k-1}), & c_k(e_{2k}) &= \frac{1}{\sqrt{2}}(e_{-2k} + e_{2k}), \\ c_k(e_j) &= e_j \text{ for } j \notin \{-2k, -2k+1, 2k-1, 2k\}. \end{aligned}$$

As in the  $SU$  and  $Sp$  settings, one passes from  $D_0$  to  $D_k$  using  $F_{(k)} = c_1^2 c_2^2 \dots c_k^2 F_{(0)}$  where the  $c_j$  are partial Cayley transforms defined by (4.3.1), as in [13]. Similarly, as in [11], the boundary of  $D_0$  is the union of the orbits  $G_0(c_1 c_2 \dots c_\ell F_{(0)})$ , but here there is no closed  $G_0$ -orbit on  $\mathcal{F}_{(k)}$ ,  $k < \infty$ , and thus no Bergman–Shilov boundary of  $D_0$ .

The calculations for  $D_0$  to be a bounded symmetric domain are essentially the same as those in Section 4.1. The result is

**Proposition 4.3.2.**  *$D_0$  is realized as the bounded domain consisting of all finitary  $Z : V_- \rightarrow V_+$  such that the matrix of  $Z$  is antisymmetric and  $I - Z^*Z \gg 0$ . In that realization the action of  $G_0$  is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \rightarrow (AZ + B)(CZ + D)^{-1}$ .*

**Corollary 4.3.3.** *The bounded symmetric domain for  $SO^*(\infty)$  is a totally geodesic submanifold of the bounded symmetric domain for  $SU(\infty, \infty)$ .*

As in Section 4.2 for  $Sp(\infty; \mathbb{R})$ , the  $G_0$ -orbit of signature  $(pos, neg, nul) = (a, b, c)$ ,  $a$  and  $c$  finite, is

$$\begin{aligned} D_{a,b,c} &= G_0(0, (F_+ + F_- + F_0), V) \text{ where} \\ F_0 &= \text{Span}\{(e_{-c} + e_c), \dots, (e_{-1} + e_1)\} \text{ (} h\text{-null)}, \\ F_+ &= \text{Span}\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (} h\text{-positive definite)}, \\ F_- &= \text{Span}\{e_{c+1}, e_{c+2}, \dots\} \text{ (} h\text{-negative definite)}, \end{aligned}$$

and every  $G_0$ -orbit on  $\mathcal{Z}_{\mathcal{F}, E}$  is one of those  $D_{a,b,c}$ . We always have  $b = \dim F_- = \infty$ . The open orbits are the case  $c = 0$  mentioned above:  $D_k = D_{k,b,0}$ .

#### 4.4 The Complex Bounded Symmetric Domain for $SO(\infty, 2)$ .

This one is more delicate because the bounded domain for  $SO(\infty, 2)$  does not sit as an easily described totally geodesic submanifold of the bounded domain for any of the  $SU(\infty, q)$ . Specifically, it is a bounded domain in a nondegenerate complex quadric in an infinite dimensional complex projective space.

We use a basis

$$(4.4.1) \quad E = \{\dots, e_{-3}, e_{-2}, e_{-1}, e_1, e_2\}$$

of  $V$ .  $G_0 = SO(\infty, 2) = SO(\infty; \mathbb{C}) \cap U(\infty, 2)$  is the connected real semisimple Lie group defined by the following hermitian form  $h$  and the symmetric bilinear form  $b$ :

$$(4.4.2) \quad h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0, h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0, b(e_i, e_j) = \delta_{i,j} \text{ for all } i, j.$$

This is a finitary change from (3.2.3) and (3.2.4). The only effect of the change is to facilitate our study of bounded domains and cycle spaces for  $SO(\infty, 2)$ .

For  $n > 0$  we have  $E_n = \{e_{-n}, e_{-n+1}, \dots, e_{-1}, e_1, e_2\}$ , the  $(n+2)$ -dimensional subspace  $V_n = \text{Span}(E_n)$  of  $V$ , the  $(n+1)$ -dimensional complex projective space  $\mathcal{P}^{n+1} = \mathcal{P}(V_n)$ , and the nondegenerate complex quadric  $\mathcal{Z}_n = \{[v] \in \mathcal{P}^{n+1} \mid b(v, v) = 0\}$ . They define the infinite dimensional complex projective space  $\mathcal{P}^\infty = \mathcal{P}(V) = \varinjlim \mathcal{P}^{n+1}$  and the nondegenerate complex quadric  $\mathcal{Z} = \{[v] \in \mathcal{P}^\infty \mid b(v, v) = 0\} = \varinjlim \mathcal{Z}_n$  in  $\mathcal{P}^\infty$ . Note that everything here is finitary. The complex group  $G = SO(\infty; \mathbb{C})$  is transitive on  $\mathcal{Z}$  because  $SO(n+2; \mathbb{C})$  is transitive on  $\mathcal{Z}_n$ .

Our bounded symmetric domain will be  $D_0 = G_0(z_0) \subset \mathcal{Z}$  where  $z_0 = [e_1 + \sqrt{-1}e_2]$ . We now look at the Harish-Chandra embedding of  $D_0$  in its holomorphic tangent space. The Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -{}^tB & D \end{pmatrix} \mid {}^tA = -A, {}^tD = -D \right\} \text{ where } A \in \mathbb{C}^{\infty \times \infty}, B \in \mathbb{C}^{\infty \times 2}, D \in \mathbb{C}^{2 \times 2}$$

and the isotropy subalgebra at  $z_0$  is the parabolic

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ -{}^tB & D \end{pmatrix} \in \mathfrak{g} \mid B = (B'', \sqrt{-1}B'') \right\} \text{ where } B'' \in \mathbb{C}^{\infty \times 1}.$$

The holomorphic tangent space to  $\mathcal{Z}$  at  $z_0$  is

$$\mathfrak{m}^+ = \left\{ \begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix} \mid B = (\sqrt{-1}B'', B'') \text{ with } B'' \in \mathbb{C}^{\infty \times 1} \right\}.$$

We view  $\mathfrak{m}^+$  as  $\mathbb{C}^\infty$  (column vectors)  $Z$  under the correspondence

$$Z \mapsto \tilde{Z} := \begin{pmatrix} 0 & Z' \\ -{}^tZ' & 0 \end{pmatrix} \text{ where } Z' = (\sqrt{-1}Z, Z) \in \mathbb{C}^{\infty \times 2}.$$

Computing as in [13], the composition  $\xi : \mathbb{C}^\infty \rightarrow \mathfrak{m}^+ \rightarrow \mathcal{Z}$  corresponding to the Harish-Chandra embedding is

$$\xi(Z) = (\exp(\tilde{Z})(z_0) = \left[ \begin{array}{c} 2\sqrt{-1}Z \\ 1+{}^tZ Z \\ \sqrt{-1}(1-{}^tZ Z) \end{array} \right] \in \mathcal{Z} \subset \mathcal{P}(V).$$

Now  $h(\xi(Z), \xi(Z)) = 2Z^* \cdot 2Z - |1 + {}^tZ Z|^2 - |1 - {}^tZ Z|^2 < 0$ , so

$$h(\xi(Z), \xi(Z)) < 0 \Leftrightarrow 1 + |{}^tZ Z|^2 - 2Z^* Z > 0 \Leftrightarrow (1 - Z^* Z)^2 > (Z^* Z)^2 - |{}^tZ Z|^2.$$

Using  $Z^* Z \geq |{}^tZ Z| \geq 0$  we take positive square roots to see

$$\{Z \in \mathbb{C}^\infty \mid h(\xi(Z), \xi(Z)) < 0\} = D'_0 \cup D'_1 \text{ (disjoint)}$$

where  $D'_0$  is the nonempty bounded domain star shaped from 0,

$$D'_0 = \{Z \in \mathbb{C}^\infty \mid 1 - Z^* Z > ((Z^* Z)^2 - |{}^tZ Z|^2)^{1/2}\},$$

and  $D'_1$  is the nonempty unbounded domain star shaped from  $\infty$ ,

$$D'_1 = \{Z \in \mathbb{C}^\infty \mid Z^* Z - 1 > ((Z^* Z)^2 - |{}^tZ Z|^2)^{1/2}\}.$$

Using Witt's Theorem on the finite dimensional approximations,  $\xi^{-1}(D_0)$  is the topological component of  $\{Z \in \mathbb{C}^\infty \mid h(\xi(Z), \xi(Z)) < 0\}$  containing 0 so  $D'_0 = \xi^{-1}(D_0)$ . We have proved

**Proposition 4.4.3.** *The bounded symmetric domain  $D_0$  for  $SO(\infty, 2)$  is given by*

$$\begin{aligned} \xi^{-1}(D_0) &= \{Z \in \mathbb{C}^\infty \mid 1 - Z^* Z > ((Z^* Z)^2 - |{}^tZ Z|^2)^{1/2}\} \\ &= \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZ Z|^2 - 2Z^* Z > 0 \text{ and } Z^* Z < 1\}. \end{aligned}$$

Shortly we will see this in terms of partial Cayley transforms, but for the moment we mention that  $D'_1 = \xi^{-1}(D_1)$  where  $z_1 = [e_1 - \sqrt{-1}e_2]$  and  $D_1 = G_0(z_1)$  is given by

$$\begin{aligned}\xi^{-1}(D_1) &= \{Z \in \mathbb{C}^\infty \mid Z^* Z - 1 > ((Z^* Z)^2 - |{}^t Z Z|^2)^{1/2}\} \\ &= \{Z \in \mathbb{C}^\infty \mid 1 + |{}^t Z Z|^2 - 2Z^* Z > 0 \text{ and } Z^* Z > 1\}.\end{aligned}$$

The action of  $G_0$  on  $D_0$  is somewhat complicated because of the quadratic term  $q : \mathbb{C}^\infty \rightarrow \mathbb{C}$  given by  $q(Z) = {}^t Z Z$ . If  $Z \in \xi^{-1}(D_0)$  the  $Z^* Z < 1$  so  $|q(Z)| < 1$ , and the formula for  $\xi(Z)$  says

$$\text{if } q = q(Z) \neq 1 \text{ then } \xi(Z) = (\exp(\tilde{Z}))(z_0) = \begin{bmatrix} \frac{2\sqrt{-1}Z}{1+q(Z)} \\ \frac{1}{1-\sqrt{-1}q(Z)} \end{bmatrix} \in \mathcal{Z} \subset \mathcal{P}(V).$$

Now, by straightforward computation,

**Proposition 4.4.4.** *The action  $g(Z) = \xi^{-1}g\xi(Z)$  of  $G_0$  on the open orbit  $D_0$  is given by*

$$\text{if } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{-1}Z}{1+q(Z)} \\ \frac{1}{1-\sqrt{-1}q(Z)} \end{pmatrix} \text{ then } g(Z) = \frac{1}{(1, \sqrt{-1})(CZ_1 + DZ_2)}(AZ_1 + BZ_2).$$

Here  $\frac{1}{(1, \sqrt{-1})(CZ_1 + DZ_2)}$  is  $1 \times 1$  and is viewed as a complex number.

Express  $V = V_+ \oplus V_-$  where  $V_+ = \text{Span}\{e_i \mid i < 0\}$  and  $V_- = \text{Span}\{e_1, e_2\}$ . Then  $h(V_+, V_-) = 0 = b(V_+, V_-)$ . Let  $[v] \in \mathcal{Z} \subset \mathcal{P}(V)$  with  $G_0([v])$  open in  $\mathcal{Z}$ , in other words with  $h(v, v) \neq 0$ . If  $\pi_-(v) = a(e_1 + \sqrt{-1}e_2) + b(e_1 - \sqrt{-1}e_2)$  then  $0 = b(v, v) = 2ab$ . Replacing  $v$  within  $[v]$  now the only possibilities are (i)  $\pi_-(v) = (e_1 + \sqrt{-1}e_2)$ , (ii)  $\pi_-(v) = (e_1 - \sqrt{-1}e_2)$  and (iii)  $v \in V_+$ . The domains  $D_0 = G_0([e_1 + \sqrt{-1}e_2])$  and  $D_1 = G_0([e_1 - \sqrt{-1}e_2])$ , so the possibilities (i) and (ii) correspond to  $D_0$  and  $D_1$ . They are equivalent under complex conjugation and each has signature  $(0, 1, 0)$ . See Remark 3.2.5. The bounded symmetric domains  $D_0$  and  $D_1$  are of tube type.

The  $G_0$ -stabilizer of  $V_+$ , which is  $SO(\infty) \times SO(2)$ , is transitive on the projective light cone in  $V_+$ ; thus the possibility (iii) corresponds to the domain  $D_2 = G_0([e_{-1} + \sqrt{-1}e_{-2}])$ , signature  $(1, 0, 0)$ . This completes the verification of

**Lemma 4.4.5.** *There are three open orbits for the action of  $SO(\infty, 2)$  on  $\mathcal{Z}$ : the two  $h$ -negative definite orbits  $D_0 = G_0([e_1 + \sqrt{-1}e_2])$  and  $D_1 = G_0([e_1 - \sqrt{-1}e_2])$ , and the  $h$ -positive definite orbit  $D_2 = G_0([e_{-1} + \sqrt{-1}e_{-2}])$ .*

Now we do this more carefully with the partial Cayley transforms. Each  $c_i(e_j) = e_j$  for  $j \notin \{-2, -1, 1, 2\}$ . In the basis  $\{e_{-2}, e_{-1}, e_1, e_2\}$  of  $\text{Span}\{e_{-2}, e_{-1}, e_1, e_2\}$ , the  $c_i$  have matrices

$$(4.4.6) \quad c_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \text{ and } c_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Thus there are six  $G_0$ -orbits on  $\mathcal{Z}$ . Their base points are

$$(4.4.7) \quad \begin{aligned} z_{0,0} = z_0 &= [e_1 + \sqrt{-1}e_2] && \text{(negative)} \\ z_{0,1} = c_1^2 z_0 &= [e_{-2} + \sqrt{-1}e_{-1}] && \text{(positive)} \\ z_{0,2} = c_1^2 c_2^2 z_0 &= [e_1 - \sqrt{-1}e_2] && \text{(negative)} \\ z_{1,1} = c_1 z_0 &= [e_{-2} + \sqrt{-1}e_{-1} + e_1 + \sqrt{-1}e_2] && \text{(isotropic)} \\ z_{1,2} = c_1 c_2^2 z_0 &= [e_{-2} - \sqrt{-1}e_{-1} - e_1 + \sqrt{-1}e_2] && \text{(isotropic)} \\ z_{2,2} = c_1 c_2 z_0 &= [e_{-2} + \sqrt{-1}e_2] && \text{(isotropic)} \end{aligned}$$

That gives 3 open orbits  $D_0 = G_0(z_{0,0})$ ,  $D_2 = G_0(z_{0,1})$  and  $D_1 = G_0(z_{0,2})$ ; it gives two intermediate orbits  $G_0(z_{1,1})$  and  $G_0(z_{1,2})$ ; and it gives one closed orbit  $G_0(z_{2,2})$ .

## 4.5 Bounded Symmetric Domains for $SL(\infty; \mathbb{R})$ and $SL(\infty; \mathbb{H})$ .

There are no complex bounded symmetric domains for these  $SL(m; \mathbb{F})$ ,  $m < \infty$ , except the unit disk in  $\mathbb{C}$ , corresponding to  $SL(2; \mathbb{R}) \cong SU(1, 1) \cong SL(1; \mathbb{H})$ . In particular there is no complex bounded symmetric domain for  $SL(\infty; \mathbb{R})$ , and there is none for  $SL(\infty; \mathbb{H})$ .

## 5 Cycles and Cycle Spaces

In the finite dimensional setting, where  $D$  is an open  $G_0$ -orbit (flag domain) in  $\mathcal{Z} = G/Q$ , a maximal compact subgroup  $K_0 \subset G_0$  has just one orbit  $Y$  on  $D$  that is a complex submanifold [12]. The  $G$ -translates of  $Y$  that are contained in  $D$  form the *cycle space*  $\mathcal{M}_D$ . That cycle space is sometimes called the *universal domain* or *crown* of the flag domain. It has many uses in harmonic analysis and algebraic geometry; see [3]. It also has remarkable complex-geometric and function-analytic properties; for example it is a contractible Stein manifold, it has an explicit geometric description, and it is the key ingredient for the double fibration transform of which one special case is the Penrose Transform. Here we extend some of the basic results on cycle spaces to an infinite dimensional setting.

### 5.1 Basic Results.

We fix an open  $G_0$ -orbit  $D$  in the complex flag manifold  $\mathcal{Z}_{E, \mathcal{F}} \cong G/Q$  with  $E$  as described in Section 3 and  $\mathcal{F}$  compatible with  $E$ . (The results of Section 3.7 show when there are open  $G_0$ -orbits in  $\mathcal{Z}_{E, \mathcal{F}}$ .) Let  $K_0$  be a maximal lim-compact subgroup of  $G_0$  and let  $K \subset G$  be its complexification. As in the finite dimensional case [3, Theorem 4.3.1],

**Theorem 5.1.1.** *There is a unique orbit  $K_0(z) \subset D$  such that  $K_0(z)$  is a complex submanifold of the flag manifold  $\mathcal{Z}_{\mathcal{F}, E}$ . Further,  $K \cap Q_z$  is a parabolic subgroup of  $K$  and  $K_0(z) = K(z) \cong K/(K \cap Q_z)$ , so  $K_0(z)$  is a complex flag manifold.*

*If  $C \subset D$  is a lim-compact complex submanifold then the following are equivalent: (i)  $C$  is a  $K_0$ -orbit, (ii)  $C$  is a  $K$ -orbit, and (iii)  $C = K_0(z)$ .*

*Proof.* The idea is to use the bases of Section 3 together with the results of [2, Section 6] in order to take a direct limit using the finite dimensional flag domain result of [3, Theorem 4.3.1].

We run through the cases of Section 3. In each case, the basis  $E$  of  $V$  is a disjoint union of finite sets  $E_\ell$  where (i) if there is a hermitian form  $h$  then the  $\text{Span}\{E_\ell\}$  are  $h$ -nondegenerate and mutually  $h$ -orthogonal, (ii) if there is a bilinear form  $b$  then the  $\text{Span}\{E_\ell\}$  are  $b$ -nondegenerate and mutually  $b$ -orthogonal as well. Further, we may assume that  $\ell$  runs over the positive integers,

Denote  $\tilde{E}_\ell = \bigcup_{k < \ell} E_k$  and  $V_\ell = \text{Span}\{\tilde{E}_\ell\}$ . In view of (i) and (ii) just above,  $G_0 = \varinjlim G_{\ell, 0}$  where  $G_{\ell, 0} = \{g \in G_0 \mid gV_\ell = V_\ell\}|_{V_\ell}$  is a finite dimensional real simple Lie group, real form of the finite dimensional complex simple Lie group  $G_\ell = \{g \in G \mid gV_\ell = V_\ell\}|_{V_\ell}$ . Further,  $Q = \varinjlim Q_\ell$  where  $Q_\ell$  is the  $G_\ell$ -stabilizer of  $\mathcal{F}$ .

We need a result of Dimitrov and Penkov [2, Proposition 6.1]. They assume that  $Q$  contains a splitting Cartan subgroup of  $G$ , but the argument is valid, as in our case, when each  $Q_\ell$  contains a splitting Cartan subgroup  $H_\ell$  of  $G_\ell$  with  $H_\ell \subset H_m$  for  $\ell \leq m$ . Denote  $\tilde{E}_\ell = \bigcup_{k \leq \ell} E_k$ . Then  $\mathcal{Z}_{\mathcal{F}, E} = \varinjlim \mathcal{Z}_{\mathcal{F} \cap V_\ell, \tilde{E}_\ell}$  where we either eliminate or ignore repetitions in the  $\mathcal{F} \cap V_\ell$ .

Since  $D$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ , the flag  $\mathcal{F}$  is nondegenerate in  $V$ , so by construction of the  $\tilde{E}_\ell$  each flag  $\mathcal{F} \cap V_\ell$  is nondegenerate in  $V_\ell$ . Thus  $G_{\ell, 0}(\mathcal{F} \cap V_\ell)$  is open in  $\mathcal{Z}_{\mathcal{F} \cap V_\ell, \tilde{E}_\ell}$  for each  $\ell$ . It follows [12, Theorem 2.12] that, for each  $\ell$ ,  $Q_\ell$  contains a fundamental Cartan subgroup  $T_{\ell, 0}$



of  $G_{\ell,0}$ . Any two fundamental Cartan subgroups of  $G_{\ell,0}$  are conjugate, and if  $k \leq \ell$  then any fundamental Cartan subgroup of  $G_{k,0}$  is contained in a fundamental Cartan subgroup of  $G_{\ell,0}$ . Thus we may assume  $T_{k,0} \subset T_{\ell,0}$  for  $k \leq \ell$ .

The fundamental Cartan  $T_{\ell,0}$  determines a maximal compact subgroup  $K_{\ell,0}$  of  $G_{\ell,0}$  such that  $T_{\ell,0} \cap K_{\ell,0}$  is a compact Cartan subgroup of  $K_{\ell,0}$ . Let  $K_\ell$  denote the complexification of  $K_{\ell,0}$ . Now  $K_{k,0} \subset K_{\ell,0}$  and  $K_k \subset K_\ell$  for  $k \leq \ell$ . Following [3, Theorem 4.3.1],  $K_{\ell,0}(\mathcal{F} \cap V_\ell)$  is the unique  $K_{\ell,0}$ -orbit in  $G_{\ell,0}(\mathcal{F} \cap V_\ell)$  that is a complex submanifold of  $\mathcal{Z}_{\mathcal{F} \cap V_\ell, \tilde{E}_\ell}$ , and  $K_{\ell,0}(\mathcal{F} \cap V_\ell) = K_\ell(\mathcal{F} \cap V_\ell)$ .

Suppose for the moment that  $K_0 = \lim_{\leftarrow} K_{\ell,0}$ . Then  $K_0(\mathcal{F})$  is the unique  $K_0$ -orbit in  $D$  that is a complex submanifold of  $\mathcal{Z}_{\mathcal{F}, E}$  and  $\overline{K_0(\mathcal{F})} = K(\mathcal{F})$ . Theorem 5.1.1 follows for this particular maximal lim-compact subgroup  $K_0$  in  $G_0$ . But any two maximal lim-compact subgroups of  $G_0$  are conjugate, so Theorem 5.1.1 follows for every choice of  $K_0$ .  $\square$

Let  $K_0$  be the maximal lim-compact subgroup of  $G_0$  constructed above in the proof of Theorem 5.1.1. We will use the notation

$$(5.1.2) \quad Y = K_0(\mathcal{F}), \quad G\{Y\} = \{g \in G \mid gY \subset D\}, \quad G_Y = \{g \in G \mid gY = Y\}, \quad \mathcal{M}'_D = G\{Y\} \cdot Y.$$

where  $Y$  is the complex  $K_0$ -orbit in the open  $G_0$ -orbit  $D \subset \mathcal{Z}_{\mathcal{F}, E}$ . We refer to  $Y$  as the **base cycle** in  $D$ . Note that the elements of  $G\{Y\}$  do not have to map  $D$  into itself;  $G\{Y\}$  simply is the set of all elements in the complex group that keep the base cycle  $Y$  inside  $D$ . Further,  $\mathcal{M}'_D := G\{Y\} \cdot Y$  is the set of all such  $G$ -translates of  $Y$ .

**Lemma 5.1.3.**  *$G\{Y\}$  is an open subset of  $G$ ,  $G_Y$  is a closed complex subgroup of  $G$ , and  $\mathcal{M}'_D = G\{Y\}/G_Y$  is an open subset of the complex manifold  $G/G_Y$ . In particular  $\mathcal{M}'_D$  is an open complex submanifold of  $G/G_Y$ .*

*Proof.* For each  $\ell$ ,  $G_Y \cap G_\ell$  is a closed complex subgroup of  $G_\ell$  and  $G\{Y\} \cap G_\ell$  is an open subset of  $G_\ell$ . It follows that  $G_Y$  is a closed complex subgroup of  $G$  and  $G\{Y\}$  is an open subset of  $G$ . Now  $G\{Y\}/G_Y$  is open in the complex homogeneous space  $G/G_Y$ .  $\square$

The complex manifold structure of  $\mathcal{M}'_D$  specifies its topology, and we define

**Definition 5.1.4.** *Let  $\mathcal{M}_D$  denote the topological component of  $Y$  in  $\mathcal{M}'_D$ . Then  $\mathcal{M}_D$  is the cycle space of the flag domain  $D$ .*

Note that  $\mathcal{M}_D$  is not always the same as the Barlet cycle space [9] from the theory of complex analytic spaces. See [3, Part IV] for the comparison. Next, we discuss several cases where we can really pin down the structure of  $\mathcal{M}_D$ .

## 5.2 Cycle Spaces for $SU(\infty, q)$ , $q \leq \infty$ .

In this section  $G_0 = SU(\infty, q)$  and its maximal lim-compact subgroup is

$$\begin{aligned} K_0 &= S(U(\infty) \times U(q)) = \lim_{p \rightarrow \infty} S(U(p) \times U(q)) \text{ if } q < \infty, \\ K_0 &= S(U(\infty) \times U(\infty)) = \lim_{r, s \rightarrow \infty} S(U(r) \times U(s)) \text{ if } q = \infty. \end{aligned}$$

This corresponds to an  $h$ -orthogonal decomposition

$$(5.2.1) \quad \begin{aligned} \mathbb{C}^{\infty, q} &= V_+ \oplus V_- \text{ where } V_+ = \text{Span}\{\dots, e_{-3}, e_{-2}, e_{-1}\}, V_- = \text{Span}\{e_1, \dots, e_q\} \text{ or} \\ \mathbb{C}^{\infty, \infty} &= V_+ \oplus V_- \text{ where } V_+ = \text{Span}\{\dots, e_{-3}, e_{-2}, e_{-1}\}, V_- = \text{Span}\{e_1, e_2, e_3, \dots\}. \end{aligned}$$

Here we use the related orthogonal basis  $E$  given by (3.1.1) and the hermitian form  $h$  of (3.1.2) that defines  $G_0$ . Let  $\mathcal{F} = (F_k)$  be a generalized flag in  $V = \mathbb{C}^{\infty, q}$  that is weakly compatible

with  $E$ . Let  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$  so that  $D = G_0(\mathcal{F}^{(1)})$  is an open  $G_0$ -orbit. Then we may assume that  $\mathcal{F}^{(1)}$  is compatible with our choice of  $E$ , so it fits the decomposition (5.2.1) in the sense that

$$(5.2.2) \quad \mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

Then  $K_0(\mathcal{F}^{(1)})$  is the unique  $K_0$ -orbit in  $D$  that is a complex submanifold of the flag manifold  $\mathcal{Z}_{\mathcal{F},E}$ . Concretely,  $K_0(\mathcal{F}^{(1)})$  is the product of “smaller” complex flag manifolds,

$$(5.2.3) \quad \begin{aligned} Y &= Y_1 \times Y_2 \text{ where} \\ Y_1 &= K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_- \end{aligned}$$

where  $\mathcal{F}^{(1)} \cap V_+$  is the generalized flag of the  $(F_k^{(1)} \cap V_+)$  and  $\mathcal{F}^{(1)} \cap V_-$  is the generalized flag of the  $(F_k^{(1)} \cap V_-)$ , ignoring repetitions. The signature sequence  $\{(a_k, b_k)\}$ , where  $h$  has signature  $(a_k, b_k, 0)$  on  $F_k^{(1)}$ , specifies the open orbit in  $\mathcal{Z}_{\mathcal{F},E}$  and the factors of  $Y$ .

As in the finite dimensional case, this shows that the  $G$ -translates of  $Y$  contained in  $D$  correspond to the decompositions  $V = W' \oplus W''$  where (i)  $W'$  is a maximal positive definite subspace such that  $V_+ \cap W'$  has finite codimension in both in  $V_+$  and in  $W'$ , and (ii)  $W''$  is a maximal negative definite subspace such that  $V_- \cap W''$  has finite codimension in both in  $V_-$  and in  $W''$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$  the correspondence depends only on  $W'$ , and if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$  it depends only on  $W''$ . Any two such decompositions  $V = W' \oplus W''$  are  $G$ -equivalent.

**Definition 5.2.4.** The *positive bounded symmetric domain*  $\mathcal{B}_E^+$  associated to  $(V, E)$  is the space of all maximal positive definite subspaces  $W' \subset V$  such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ . The *negative bounded symmetric domain*  $\mathcal{B}_E^-$  associated to  $(V, E)$  is the space of all maximal negative definite subspaces  $W'' \subset V$  such that  $W'' \cap V_-$  has finite codimension in both  $W''$  and  $V_-$ .

As constructed, each element  $W' \in \mathcal{B}_E^+$  is in the  $G$ -orbit of  $V_+$ . Relative to the basis  $E$  we look at  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$  such that  $gW' \in \mathcal{B}_E^+$ , in other words such that the column span of  $\begin{pmatrix} A \\ C \end{pmatrix}$  is positive definite. The column span is preserved under right multiplication by  $A$ , so the positive definite condition is  $\begin{pmatrix} I \\ -CA^{-1} \end{pmatrix}^* \cdot \begin{pmatrix} I \\ CA^{-1} \end{pmatrix} \gg 0$ . In other words  $gW' \in \mathcal{B}_E^+$  simply means that  $gW'$  is the column span of an infinite matrix  $\begin{pmatrix} I \\ Z_1 \end{pmatrix}$  such that  $I - Z_1^* Z_1 \gg 0$ . Similarly  $gW'' \in \mathcal{B}_E^-$  simply means that  $gW''$  is the column span of an infinite matrix  $\begin{pmatrix} Z_2 \\ I \end{pmatrix}$  such that  $I - Z_2 Z_2^* \gg 0$ . The distinction is that the  $G$ -stabilizer of  $V_+ \in \mathcal{B}_E^+$  is the parabolic  $P$  consisting of all  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ , while the  $G$ -stabilizer of  $V_- \in \mathcal{B}_E^-$  is the opposite parabolic  ${}^tP = P^{opp}$  consisting of all  $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ . Thus they have conjugate complex structures:  $\mathcal{B}_E^+$  has holomorphic tangent space represented by the matrices  $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{g}$  while the holomorphic tangent space of  $\mathcal{B}_E^-$  is represented by the  $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ .

Reformulating this,

**Lemma 5.2.5.** Suppose that  $G_0 = SU(\infty, q)$ ,  $q \leq \infty$ . Then the positive bounded symmetric domain associated to the triple  $(V, G_0, E)$  is  $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty \times q} \mid I - Z_1^* Z_1 \gg 0\}$  in  $G/P$ , and the negative bounded symmetric domain for  $(V, G_0, E)$  is the complex conjugate domain  $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{q \times \infty} \mid I - Z_2^* Z_2 \gg 0\}$  in  $G/P^{opp}$ .

The action of  $G_0$  on these bounded symmetric domains is described in Section 4.1.

Now we are ready to prove the following theorem.

**Theorem 5.2.6.** Let  $G_0 = SU(\infty, q)$  with  $q \leq \infty$ . Let  $D$  be an open  $G_0$ -orbit  $G(\mathcal{F}^{(1)})$  in  $\mathcal{Z}_{\mathcal{F},E}$ . In the notation of (5.2.1), the positive definite bounded symmetric domain  $\mathcal{B}_E^+$  for  $(V, G_0, E)$  is

the set of all positive definite  $G$ -translates of  $V_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_E^-$  for  $(V, G_0, E)$  is the set of all negative definite  $G$ -translates of  $V_-$ . The  $\mathcal{B}_E^\pm$  are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.

- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+$ .
- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^-$ .
- If some  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+ \times \mathcal{B}_E^-$ .

*Proof.* Directly from Definition 5.2.4,  $gV_+$  is  $h$ -positive definite if and only if  $gV_+ \in \mathcal{B}_E^+$ ,  $gV_-$  is  $h$ -negative definite if and only if  $gV_- \in \mathcal{B}_E^-$ , and both properties hold for  $gV_\pm$  if and only if  $(gV_+, gV_-) \in \mathcal{B}_E^+ \times \mathcal{B}_E^-$ .

First suppose that  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$ ,  $g \in G$  and  $k \in K_0$ . Note  $kV_+ = V_+$ . If  $gV_+$  is positive definite then  $gk(\mathcal{F}^{(1)}) \in D$  because  $gk(\mathcal{F}^{(1)})$  is nondegenerate and  $D$  is the only open  $G_0$ -orbit in  $\mathcal{Z}_{\mathcal{F}, E}$  consisting of positive definite subspaces. Thus  $gY \subset D$ , in other words  $gY \in \mathcal{M}'_D$ . Conversely if  $gY \in \mathcal{M}'_D$ , so  $gY \subset D$ , then  $gY$  consists of positive definite subspaces. If  $0 \neq F^{(1)} \in \mathcal{F}^{(1)}$  then  $\text{Span } K_0(F^{(1)}) = V_+$ , so  $\text{Span } gY = gV_+$  is positive definite. Now  $gY \in \mathcal{M}'_D$  if and only if  $gV_+ \in \mathcal{B}_E^+$ .

Similarly, if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$  and  $g \in G$  then  $gY \in \mathcal{M}'_D$  if and only if  $gV_- \in \mathcal{B}_E^-$ .

In the general case  $\mathcal{F}^{(1)} \cap V_+ \neq \mathcal{F}^{(1)} \neq \mathcal{F}^{(1)} \cap V_-$  the arguments just above show that  $gV_+$  is positive definite if and only if  $gK_0(\mathcal{F}^{(1)} \cap V_+)$  consists of positive definite subspaces; and  $gV_-$  is negative definite if and only if  $gK_0(\mathcal{F}^{(1)} \cap V_-)$  consists of negative definite subspaces. Thus  $gY \in \mathcal{M}'_D$  if and only if  $gV_+$  is positive definite and  $gV_-$  is negative definite, in other words if and only if  $(gV_+, gV_-) \in \mathcal{B}_E^+ \times \mathcal{B}_E^-$ .

In all three cases we note that  $\mathcal{M}'_D$  is connected, so  $\mathcal{M}'_D = \mathcal{M}_D$ .

Finally,  $h$ -orthocomplementation is antiholomorphic and interchanges  $\mathcal{B}_E^+$  with  $\mathcal{B}_E^-$ .  $\square$

### 5.3 Cycle Spaces for $Sp(\infty; \mathbb{R})$ .

The case  $G_0 = Sp(\infty; \mathbb{R}) = Sp(\infty; \mathbb{C}) \cap U(\infty, \infty)$  differs from the  $SU(\infty, q)$  cases mainly in that we use  $b$ -isotropic flags where  $b$  is the antisymmetric bilinear form that defines  $Sp(\infty; \mathbb{C})$ . Specifically, we use the basis and forms described in Section 3, given by (3.4.1) and (3.5.1), where  $b$  defines  $Sp(\infty; \mathbb{C})$  and  $h$  defines  $U(\infty, \infty)$ .

The maximal lim-compact subgroups  $K_0$  of  $G_0 = Sp(\infty; \mathbb{R})$  is the  $U(\infty)$  constructed as follows. Relative to  $h$ ,

$$(5.3.1) \quad V = V_+ \oplus V_- \text{ where } V_+ = \text{Span}\{\dots, e_{-3}, e_{-2}, e_{-1}\} \text{ and } V_- = \text{Span}\{e_1, e_2, e_3, \dots\}.$$

The maximal lim-compact subgroup of  $U(\infty, \infty)$  is  $U(V_+) \times U(V_-) = U(\infty) \times U(\infty)$ , and  $K_0$  is the subgroup  $G_0 \cap (U(\infty) \times U(\infty)) \cong U(\infty)$ . In the ordered basis  $\{e_{-1}, e_{-2}, \dots; e_1, e_2, \dots\}$  it would be diagonally embedded in  $U(\infty) \times U(\infty)$ .

Let  $\mathcal{F}$  be a  $b$ -isotropic generalized flag compatible with  $E$ . Let  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$  so that  $D = G_0(\mathcal{F}^{(1)})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ . Again, we may assume that  $\mathcal{F}^{(1)}$  is compatible with  $E$ , in other words, it fits the splitting (5.3.1) in the sense that

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

In particular  $\mathcal{F}^{(1)}$  is  $h$ -nondegenerate, corresponding to the fact that  $D = G_0(\mathcal{F})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ , and  $K_0(\mathcal{F}^{(1)})$  is the unique  $K_0$ -orbit in  $D$  that is a complex submanifold of  $\mathcal{Z}_{\mathcal{F}, E}$ .

**Lemma 5.3.2.** Define  $\mathcal{F}^{(1)} \cap V_+ = (F_k^{(1)} \cap V_+)$  and  $\mathcal{F}^{(1)} \cap V_- = (F_k^{(1)} \cap V_-)$ , and spaces  $W_+ = \bigcup_k (F_k^{(1)} \cap V_+)$  and  $W_- = \bigcup_k (F_k^{(1)} \cap V_-)$ . Then the complex lim-compact group orbit  $Y = K_0(\mathcal{F}^{(1)})$  is the subvariety of

$$\begin{aligned} \tilde{Y} &= Y_1 \times Y_2 \text{ where } Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_- \end{aligned}$$

defined by  $b(k_1 W_+, k_2 W_-) = 0$  for  $k_1, k_2 \in K_0$ . The signature sequence  $\{(a_k, b_k)\}$ , where  $h$  has signature  $(a_k, b_k, 0)$  on  $F_k^{(1)}$ , specifies the open orbit in  $\mathcal{Z}_{\mathcal{F}, E}$  and the factors of  $\tilde{Y}$ .

*Proof.* The projections  $r_1 : K_0 \rightarrow U(V_+)$  and  $r_2 : K_0 \rightarrow U(V_-)$  are isomorphisms. Define  $\mu : V \rightarrow V$  by  $\mu(e_i) = e_{-i}$  and  $\mu(e_{-i}) = -e_i$  for  $i > 0$ . Since  $\mathcal{F}^{(1)}$  is  $b$ -isotropic and compatible with  $E$ , each  $F_k^{(1)}$  is spanned by a subset  $S_k \subset E$  that never contains a pair  $\{e_i, e_{-i}\}$ . Thus each  $(\mathcal{F}_k^{(1)} \cap V_+) + \mu(\mathcal{F}_k^{(1)} \cap V_+)$  is  $b$ -nondegenerate and  $h$ -nondegenerate, and is orthogonal to  $(\mathcal{F}_k^{(1)} \cap V_-) + \mu(\mathcal{F}_k^{(1)} \cap V_-)$  relative to both  $b$  and  $h$ . Now the action of  $r_1(K_0)$  on  $(\mathcal{F}_k^{(1)} \cap V_+) + \mu(\mathcal{F}_k^{(1)} \cap V_+)$  and the action of  $r_2(K_0)$  on  $(\mathcal{F}_k^{(1)} \cap V_-) + \mu(\mathcal{F}_k^{(1)} \cap V_-)$  only involve disjoint subsets of  $S_k \cup -S_k$ . Thus  $Y \subset \tilde{Y}$  and  $b(k_1 W_+, k_2 W_-) = 0$  for  $k_1, k_2 \in K_0$ .

Conversely, if  $(k_1(\mathcal{F}_k^{(1)} \cap V_+), k_2(\mathcal{F}_k^{(1)} \cap V_+)) \in Y$ , so it has form  $(k(\mathcal{F}_k^{(1)} \cap V_+), k(\mathcal{F}_k^{(1)} \cap V_+))$ , then  $k_1 W_+ = k W_+ \perp_b k W_- = k_2 W_-$ . Given  $k W_+ \perp_b k W_-$ ,  $K_0$  moves  $(\mathcal{F}_k^{(1)} \cap V_+)$  freely within  $V_+ \cap (W_-)^\perp$  and moves  $(\mathcal{F}_k^{(1)} \cap V_-)$  freely within  $V_+ \cap (W_+)^\perp$ . That proves the first assertion. The signature sequence assertion is contained in Theorem 3.7.2.  $\square$

These considerations show that the  $G$ -translates of  $Y$  contained in  $D$  correspond to decompositions  $V = W' \oplus W''$  where (i)  $W'$  and  $W''$  are maximal  $b$ -isotropic subspaces of  $V$ , (ii)  $W'$  is a maximal  $h$ -positive definite subspace such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ , and (iii)  $W''$  is a maximal  $h$ -negative definite subspace such that  $W'' \cap V_-$  has finite codimension in both  $W''$  and  $V_-$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$  the correspondence depends only on  $W'$ , and if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$  it depends only on  $W''$ . Any two such decompositions  $V = W' \oplus W''$  are  $G$ -equivalent.

**Definition 5.3.3.** The positive bounded symmetric domain  $\mathcal{B}_E^+$  associated to  $(V, b, E)$  is the space of all maximal  $b$ -isotropic  $h$ -positive definite subspaces  $W' \subset V$  such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ . The negative bounded symmetric domain  $\mathcal{B}_E^-$  associated to  $(V, b, E)$  is the space of all maximal  $b$ -isotropic  $h$ -negative definite subspaces  $W'' \subset V$  such that  $W'' \cap V_-$  has finite codimension in both  $W''$  and  $V_-$ .

As constructed, each element  $W' \in \mathcal{B}_E^+$  is in the  $G$ -orbit of  $V_+$ . Relative to the basis  $E$  we look at  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$  such that  $gW' \in \mathcal{B}_E^+$ , in other words such that the column span of  $\begin{pmatrix} A \\ C \end{pmatrix}$  is  $h$ -positive definite. That span is  $b$ -isotropic by definition of  $G$ , and the column span is preserved under right multiplication by  $A$ . Let  $Z_1 = CA^{-1}$ . Then the  $h$ -positive definite condition is  $\begin{pmatrix} I \\ Z_1 \end{pmatrix}^* \cdot \begin{pmatrix} I \\ Z_1 \end{pmatrix} \gg 0$ , in other words  $I - Z_1^* Z_1 \gg 0$ . Let  $Z_1 = (z_{i,j})$  where  $i, j > 0$ . The column span of  $\begin{pmatrix} I \\ Z_1 \end{pmatrix}$  has basis consisting of the  $z_j := e_{-j} + \sum_{i>0} z_{i,j} e_i$ . Compute  $b(z_j, z_\ell) = z_{j,\ell} - z_{\ell,j}$ . So the  $b$ -isotropic condition is  $Z_1 = {}^t Z_1$ . In other words  $gW' \in \mathcal{B}_E^+$  simply means that  $gW'$  is the column span of an infinite matrix  $\begin{pmatrix} I \\ Z_1 \end{pmatrix}$  such that  $I - Z_1^* Z_1 \gg 0$  and  $Z_1$  is symmetric.

Similarly  $gW'' \in \mathcal{B}_E^-$  simply means that  $gW''$  is the column span of an infinite matrix  $\begin{pmatrix} Z_2 \\ I \end{pmatrix}$  such that  $I - Z_2 Z_2^* \gg 0$  and  $Z_2$  is symmetric. The distinction is that the  $G$ -stabilizer of  $V_+ \in \mathcal{B}_E^+$  is the parabolic  $P$  consisting of all  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$  in  $\mathfrak{g}$  while the  $G$ -stabilizer of  $V_- \in \mathcal{B}_E^-$  is the opposite parabolic  ${}^t P = P^{opp}$  consisting of all  $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$  in  $\mathfrak{g}_C$ . Thus they have conjugate complex structures:  $\mathcal{B}_E^+$  has holomorphic tangent space represented by the matrices  $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$  with

$C$  symmetric while the holomorphic tangent space of  $\mathcal{B}_E^-$  is represented by the  $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$  with  $B$  symmetric.

Reformulating this,

**Lemma 5.3.4.** *Let  $G_0 = Sp(\infty; \mathbb{R})$ . Then the positive bounded symmetric domain associated to the triple  $(V, G_0, E)$  is  $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty \times \infty} \mid I - Z_1^* Z_1 \gg 0 \text{ and } Z_1 = {}^t Z_1\}$  in  $G/P$ , and the negative bounded symmetric domain for  $(V, G_0, E)$  is the complex conjugate domain  $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{\infty \times \infty} \mid I - Z_2^* Z_2 \gg 0 \text{ and } Z_1 = {}^t Z_1\}$  in  $G/P^{opp}$ .*

The action of  $G_0$  on these bounded symmetric domains is described in Section 4.2.

In any  $K_0$ -invariant Riemannian metric on  $\tilde{Y}$ ,  $Y_1$  and  $Y_2$  are the factors in the de Rham decomposition. The spaces  $k(F_\ell^{(1)} \cap V_+)$  of the elements of  $Y_1$  generate  $V_+$  (or are zero), so either  $Y$  determines  $Y_1$  determines  $V_+$ , or the  $F_\ell^{(1)} \cap V_+ = 0$ . Similarly either  $Y$  determines  $V_-$ , or the  $F_\ell^{(1)} \cap V_- = 0$ . Now apply  $g^{-1}$  whenever  $g \in G\{Y\}$  to see that  $gY$  determines  $gV_+$  or  $gV_-$  or both, as appropriate. Exactly as in the proof of Theorem 5.2.6 we arrive at the following structure theorem.

**Theorem 5.3.5.** *Let  $G_0 = Sp(\infty; \mathbb{R})$  and let  $D$  be an open  $G_0$ -orbit  $G(\mathcal{F}^{(1)})$  in  $\mathcal{Z}_{\mathcal{F}, E}$ . In the notation of (5.3.1), the positive definite bounded symmetric domain  $\mathcal{B}_E^+$  for  $(V, G_0, E)$  is the set of all positive definite  $G$ -translates of  $V_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_E^-$  for  $(V, G_0, E)$  is the set of all negative definite  $G$ -translates of  $V_-$ . The  $\mathcal{B}_E^\pm$  are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

*If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+$ .*

*If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^-$ .*

*If some  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+ \times \mathcal{B}_E^-$ .*

## 5.4 Cycle Spaces for $SO^*(\infty)$ .

The case  $SO^*(\infty) = SO(\infty; \mathbb{C}) \cap U(\infty, \infty)$  is very similar to the case of  $Sp(\infty; \mathbb{R})$ . The main difference is that the bilinear form  $b$  is symmetric rather than antisymmetric. Concretely, we have

$$E = \{\dots, e_{-k}, e_{-k+1}, \dots, e_{-1}; e_1, \dots, e_{k-1}, e_k, \dots\}, \text{ ordered basis of } V;$$

$$b(e_i, e_j) = \delta_{i+j, 0}, h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

Again we use the  $h$ -orthogonal splitting

$$(5.4.1) \quad V = V_+ \oplus V_- \text{ where } V_+ = \text{Span}\{\dots, e_{-3}, e_{-2}, e_{-1}\} \text{ and } V_- = \text{Span}\{e_1, e_2, e_3, \dots\}.$$

The maximal lim-compact subgroup of  $U(\infty, \infty)$  is  $U(V_+) \times U(V_-) = U(\infty) \times U(\infty)$ . Exactly as in the  $Sp(\infty; \mathbb{R})$  case,  $K_0$  is the subgroup  $G_0 \cap (U(\infty) \times U(\infty)) \cong U(\infty)$ . In the ordered basis  $\{e_{-1}, e_{-2}, \dots; e_1, e_2, \dots\}$  it would be diagonally embedded in  $U(\infty) \times U(\infty)$ .

Let  $\mathcal{F}$  be a  $b$ -isotropic generalized flag compatible with  $E$ . Let  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E}$  such that  $D = G_0(\mathcal{F}^{(1)})$  is open in  $\mathcal{Z}_{\mathcal{F}, E}$ . We may assume that  $\mathcal{F}$  is compatible with  $E$ , so

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

In particular  $\mathcal{F}^{(1)}$  is  $h$ -nondegenerate and  $K_0(\mathcal{F}^{(1)})$  is the unique  $K_0$ -orbit in  $D$  that is a complex submanifold of  $\mathcal{Z}_{\mathcal{F}, E}$ . With no nontrivial change, the proof of Lemma 5.3.2 also proves

**Lemma 5.4.2.** Define  $\mathcal{F}^{(1)} \cap V_+ = (F_k^{(1)} \cap V_+)$  and  $\mathcal{F}^{(1)} \cap V_- = (F_k^{(1)} \cap V_-)$ , and spaces  $W_+ = \bigcup_k (F_k^{(1)} \cap V_+)$  and  $W_- = \bigcup_k (F_k^{(1)} \cap V_-)$ . Then the complex lim-compact group orbit  $Y = K_0(\mathcal{F}^{(1)})$  is the subvariety of

$$\begin{aligned} \tilde{Y} &= Y_1 \times Y_2 \text{ where } Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_- \end{aligned}$$

defined by  $b(k_1 W_+, k_2 W_-) = 0$  for  $k_1, k_2 \in K_0$ . The signature sequence  $\{(a_k, b_k)\}$ , and (where relevant – see Remark 3.2.5) the orientation, specifies the open orbit in  $\mathcal{Z}_{\mathcal{F}, E}$  and the factors of  $\tilde{Y}$ .

Now the  $G$ -translates of  $Y$  contained in  $D$  correspond to decompositions  $V = W' \oplus W''$  where (i)  $W'$  and  $W''$  are maximal  $b$ -isotropic subspaces of  $V$ , (ii)  $W'$  is a maximal  $h$ -positive definite subspace such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ , and (iii)  $W''$  is a maximal  $h$ -negative definite subspace such that  $W'' \cap V_-$  has finite codimension in both  $W''$  and  $V_-$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$  the correspondence depends only on  $W'$ , and if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$  it depends only on  $W''$ . Any two such decompositions  $V = W' \oplus W''$  are  $G$ -equivalent.

**Definition 5.4.3.** The positive bounded symmetric domain  $\mathcal{B}_E^+$  associated to  $(V, b, E)$  consists of all maximal  $b$ -isotropic  $h$ -positive definite subspaces  $W' \subset V$  such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ . The negative bounded symmetric domain  $\mathcal{B}_E^-$  associated to  $(V, b, E)$  consists of all maximal  $b$ -isotropic  $h$ -negative definite subspaces  $W'' \subset V$  such that  $W'' \cap V_-$  has finite codimension in both  $W''$  and  $V_-$ .

Let  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$  relative to  $E$ , such that  $gW' \in \mathcal{B}_E^+$ . Computing as for  $Sp(\infty; \mathbb{R})$ , with  $Z_1 = CA^{-1}$ , we see that  $gW' \in \mathcal{B}_E^+$  if and only if  $gW'$  is the column span of an infinite matrix  $\begin{pmatrix} I \\ Z_1 \end{pmatrix}$  such that  $I - Z_1^* Z_1 \gg 0$  and  $Z_1$  is antisymmetric. Similarly  $gW'' \in \mathcal{B}_E^-$  if and only if  $gW''$  is the column span of an infinite matrix  $\begin{pmatrix} Z_2 \\ I \end{pmatrix}$  such that  $I - Z_2 Z_2^* \gg 0$  and  $Z_2$  is antisymmetric. The  $G$ -stabilizer of  $V_+ \in \mathcal{B}_E^+$  is the parabolic  $P$  consisting of all  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$  in  $\mathfrak{g}_{\mathbb{C}}$ , and the  $G$ -stabilizer of  $V_- \in \mathcal{B}_E^-$  is the opposite parabolic  ${}^tP = P^{opp}$  consisting of all  $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$  in  $\mathfrak{g}_{\mathbb{C}}$ . Thus they have conjugate complex structures:  $\mathcal{B}_E^+$  has holomorphic tangent space represented by the matrices  $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$  with  $C$  antisymmetric while the holomorphic tangent space of  $\mathcal{B}_E^-$  is represented by the matrices  $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$  with  $B$  antisymmetric.

Reformulating this,

**Lemma 5.4.4.** Let  $G_0 = SO^*(\infty)$ . Then the positive bounded symmetric domain associated to the triple  $(V, G_0, E)$  is  $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty \times \infty} \mid I - Z_1^* Z_1 \gg 0 \text{ and } Z_1 + {}^t Z_1 = 0\}$  in  $G/P$ , and the negative bounded symmetric domain for  $(V, G_0, E)$  is the complex conjugate domain  $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{\infty q \times \infty} \mid I - Z_2^* Z_2 \gg 0 \text{ and } Z_1 + {}^t Z_1 = 0\}$  in  $G/P^{opp}$ .

The action of  $G_0$  on these bounded symmetric domains is described in Section 4.3.

Arguing just as for Theorems 5.2.6 and 5.3.5, we arrive at the following structure theorem.

**Theorem 5.4.5.** Let  $G_0 = SO^*(\infty)$  and let  $D$  be an open  $G_0$ -orbit  $G(\mathcal{F}^{(1)})$  in  $\mathcal{Z}_{\mathcal{F}, E}$ . In the notation of (5.4.1), the positive definite bounded symmetric domain  $\mathcal{B}_E^+$  for  $(V, G_0, E)$  is the set of all positive definite  $G$ -translates of  $V_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_E^-$  for  $(V, G_0, E)$  is the set of all negative definite  $G$ -translates of  $V_-$ . The  $\mathcal{B}_E^{\pm}$  are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.

- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+$ .
- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^-$ .
- If some  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_E^+ \times \mathcal{B}_E^-$ .

## 5.5 Cycle Spaces for $SO(\infty, 2)$ .

Now we come to the rather delicate case  $G_0 = SO(\infty, 2)$ , where the lim-compact dual of the complex bounded symmetric domain is a nondegenerate quadric in a complex projective space. We specify  $G_0$  by the basis (4.4.1) and the forms (4.4.2). Let

$$\begin{aligned} V_{\text{even}} &= \text{Span}(\{e_{2k} + \sqrt{-1}e_{2k+1} \mid k < 0\} \cup \{e_1 - \sqrt{-1}e_2\}), \\ V_{\text{odd}} &= \text{Span}(\{e_{2k} - \sqrt{-1}e_{2k+1} \mid k < 0\} \cup \{e_1 + \sqrt{-1}e_2\}). \end{aligned}$$

They are maximal  $b$ -isotropic subspaces of  $V$ , paired by  $b(e_j + \sqrt{-1}e_{j+1}, e_j - \sqrt{-1}e_{j+1}) = 2$ . This basis  $E$  leads to the same splitting of  $V$  as the one based on (3.1.1):

$$(5.5.1) \quad V = V_+ \oplus V_- \text{ where } V_+ = \text{Span}\{\dots, e_{-3}, e_{-2}, e_{-1}\} \text{ and } V_- = \text{Span}\{e_1, e_2\}.$$

We denote

$$\mathcal{P}^\infty \text{ is the projective space } \mathcal{P}(V) \text{ and } \mathcal{Z} \text{ is the quadric } b(v, v) = 0 \text{ in } \mathcal{P}^\infty.$$

The maximal lim-compact subgroup of  $G_0$  is  $K_0 = SO(V_+) \times SO(V_-) = SO(\infty) \times SO(2)$ . The complex  $K_0$ -orbits within the open  $G_0$ -orbits on  $\mathcal{Z}$  (from Lemma 4.4.5 and (4.4.7)) are

$$(5.5.2) \quad \begin{aligned} \text{in } D_0 &= G_0([e_1 + \sqrt{-1}e_2]) : K_0([e_1 + \sqrt{-1}e_2]) = (\text{single point } [e_1 + \sqrt{-1}e_2]), \\ \text{in } D_1 &= G_0([e_1 - \sqrt{-1}e_2]) : K_0([e_1 - \sqrt{-1}e_2]) = (\text{single point } [e_1 - \sqrt{-1}e_2]), \\ \text{in } D_2 &= G_0([e_{-2} + \sqrt{-1}e_{-1}]) : K_0([e_{-2} + \sqrt{-1}e_{-1}]) = \mathcal{Z} \cap \mathcal{P}(V_+) \text{ quadric in } \mathcal{P}(V_+). \end{aligned}$$

**Definition 5.5.3.** The *positive bounded symmetric domain*  $\mathcal{B}_{E'}^+$ , associated to  $(V, b, E')$  consists of all maximal  $b$ -isotropic  $h$ -positive definite subspaces  $W' \subset V$  such that  $W' \cap V_+$  has finite codimension in both  $W'$  and  $V_+$ . Those subspaces have codimension 2 in  $V$ . The *negative bounded symmetric domain*  $\mathcal{B}_{E'}^-$ , associated to  $(V, b, E')$  consists of all maximal  $b$ -isotropic  $h$ -negative definite subspaces  $W'' \subset V$ . (Since  $\dim W'' = 2 = \dim V_-$  the finite codimension condition is automatic.)

Now more generally let  $\mathcal{F} = (F_k)$  be an isotropic generalized flag in  $V$  that is weakly compatible with  $E'$ . Let  $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F}, E'}$  for which  $D = G_0(\mathcal{F}^{(1)})$  is an open  $G_0$ -orbit. We may assume that  $\mathcal{F}^{(1)}$  is compatible with our choice of  $E'$ , so it fits the decomposition (5.4.1) as before:

$$(5.5.4) \quad \mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

Then  $K_0(\mathcal{F}^{(1)})$  is the unique  $K_0$ -orbit in  $D$  that is a complex submanifold of the flag manifold  $\mathcal{Z}_{\mathcal{F}, E'}$ . Somewhat trivially,  $K_0(\mathcal{F}^{(1)})$  is the product of “smaller” complex flag manifolds,

$$(5.5.5) \quad \begin{aligned} Y &= Y_1 \times Y_2 \text{ where} \\ Y_1 &= K_0(\mathcal{F}^{(1)} \cap V_+) = SO(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap V_-) = SO(2)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_- \end{aligned}$$

where  $\mathcal{F}^{(1)} \cap V_+ = ((F_k^{(1)} \cap V_+))$  and  $\mathcal{F}^{(1)} \cap V_- = ((F_1^{(1)} \cap V_-))$ . The signature sequence  $\{(a_k, b_k)\}$ , where  $h$  has signature  $(a_k, b_k, 0)$  on  $F_k^{(1)}$ , specifies the open orbit in  $\mathcal{Z}_{\mathcal{F}, E'}$  and the factors of  $Y$ .

If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$ , in other words  $D = D_2$  and the cycles are of the form  $K_0(gV_+)$  with  $g \in G$ , then  $M_D$  consists of the maximal  $b$ -isotropic  $h$ -positive definite subspaces of  $V$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ , in other words  $D = D_0$  or  $D = D_1$  and the cycles are single points, then  $M_D$  consists of the maximal  $b$ -isotropic  $h$ -negative definite subspaces of  $V$ . If  $\mathcal{F}^{(1)} \cap V_+ \neq \mathcal{F}^{(1)} \neq \mathcal{F}^{(1)} \cap V_-$  then  $M_D$  is the product. Thus

**Lemma 5.5.6.** *Let  $G_0 = SO(\infty, 2)$ . Then the positive bounded symmetric domain associated to the triple  $(V, G_0, E')$  is  $\mathcal{B}_{E'}^+ \cong \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZZ|^2 - 2Z^*Z > 0 \text{ and } Z^*Z < 1\}$  in  $G/P$ , and the negative bounded symmetric domain for  $(V, G_0, E')$  is the complex conjugate domain  $\mathcal{B}_{E'}^- \cong \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZZ|^2 - 2Z^*Z > 0 \text{ and } Z^*Z > 1\}$  in  $G/P^{opp}$ .*

The action of  $G_0$  on these bounded symmetric domains is described in Section 4.4.

The argument for Theorem 5.2.6 remains valid here, with one small modification. Recall Lemma 4.4.5 and (4.4.7). There is just one open orbit  $D_2 = G_0([e_{-1} + \sqrt{-1}e_{-2}])$  consisting of  $h$ -positive definite subspaces, but there are two orbits,  $D_0 = G_0([e_1 + \sqrt{-1}e_2])$  and  $D_1 = G_0([e_1 - \sqrt{-1}e_2])$ , consisting of negative definite subspaces. These last two are related by complex conjugation of  $V$  over the real span of  $E$ . Suppose that  $D$  is either  $D_0$  or  $D_2$ , that  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ , and that  $gV_-$  is negative definite. Then  $gY \subset (D_0 \cup D_1)$ . As  $gY$  is connected, either  $gY \subset D_0$  or  $gY \subset D_1$ . Thus  $gY \in \mathcal{M}'_D$ , and  $gY \in \mathcal{M}_D$  just when  $gY \subset D$ . With this adjustment the proof of Theorem 5.2.6 holds here, and the result is

**Theorem 5.5.7.** *Let  $G_0 = SO(\infty, 2)$  and let  $D$  be an open  $G_0$ -orbit  $G(\mathcal{F}^{(1)})$  in  $\mathcal{Z}_{\mathcal{F}, E'}$ . In the notation of (5.4.1), the positive definite bounded symmetric domain  $\mathcal{B}_{E'}^+$  for  $(V, G_0, E')$  is the set of all positive definite  $G$ -translates of  $V_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_{E'}^-$  for  $(V, G_0, E')$  is the set of all negative definite  $G$ -translates of  $V_-$ . The  $\mathcal{B}_{E'}^\pm$  are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_{E'}^+$ .*
- If every  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_{E'}^-$ .*
- If some  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is holomorphically diffeomorphic to  $\mathcal{B}_{E'}^+ \times \mathcal{B}_{E'}^-$ .*

## 6 Real and Quaternionic Domains and Cycle Spaces

In Section 4 we worked out the structure of finitary complex bounded symmetric domains, and in Section 5 we applied those results to obtain the structure of cycle spaces on corresponding flag domains. In this section we develop a variation on those results for particular real and quaternionic flag manifolds and cycle spaces based on the groups  $SO(\infty, q)$  and  $Sp(\infty, q)$ ,  $q \leq \infty$ . Those groups provide real and quaternionic analogs of the complex domains of  $SU(\infty, q)$ . The methods and results are similar to those of Section 4.1, Section 5.2, and the last part of [10].

### 6.1 The Real Bounded Symmetric Domain for $SO(\infty, q)$ .

In Section 4.1 we looked at the bounded domain of maximal negative definite subspaces of  $(V, h)$  contained in  $\mathcal{Z}_{\mathcal{F}, E}$ , where  $V$  has basis  $E$  given by (3.1.1) and where the hermitian form  $h$  is given by (3.1.2). We studied it as an  $SU(\infty, q)$ -orbit on the complex Grassmann manifold of  $q$ -dimensional subspaces of  $V$  weakly compatible with  $E$ . Here we look at the real analog, the (real – not complex) bounded symmetric domain of maximal negative definite subspaces of  $(V_0, b)$  where  $V_0$  is the real span of  $E$  and the symmetric bilinear form  $b$  is the restriction of  $h$  to  $V_0$ . Then we use it to describe real cycle spaces for open orbits on the corresponding real flag manifolds.

We consider the real group  $G_0 = SO(\infty, q)$ ,  $q \leq \infty$  and the flag  $\mathcal{F} = (0, F, V_0)$  where  $F = \text{Span}_{\mathbb{R}}\{e_i \mid i > 0\}$ . View  $G_0$  as a closed subgroup of  $G := SL(\infty + q; \mathbb{R})$ . That gives us the real flag manifold

$$(6.1.1) \quad \mathfrak{X}_{\mathcal{F}, E} = \{ \text{subspaces } F^{(1)} \subset V_0 \mid (0, F^{(1)}, V) \text{ is } E\text{-commensurable to } \mathcal{F} \} = G(\mathcal{F})$$



where the second equality follows as in the argument of Lemma 2.2.3. Note that  $\mathfrak{X}_{\mathcal{F},E}$  is a real Grassmann manifold. The domain of interest to us in this context is

$$(6.1.2) \quad D_0 = \{\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in \mathfrak{X}_{\mathcal{F},E} \mid F^{(1)} \text{ maximal negative definite subspace of } V_0\}.$$

If  $\tau : V \rightarrow V$  denotes complex conjugation of  $V$  over  $V_0$  then the domain  $D_0$  of (6.1.2) can be identified with the fixed point set of  $\tau$  on the complex Grassmannian of Section 4.1.

We use the  $b$ -orthogonal decomposition  $V_0 = (V_0)_+ \oplus (V_0)_-$  where  $(V_0)_+ = \text{Span}_{\mathbb{R}}\{e_i \mid i < 0\}$  and  $(V_0)_- = \text{Span}_{\mathbb{R}}\{e_i \mid i > 0\}$ . Consider the corresponding  $b$ -orthogonal projections  $\pi_{\pm}$ . The kernel of  $\pi_-$  is  $b$ -positive definite so it has zero intersection with  $F^{(1)}$  for any  $\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in D_0$ . Thus  $\pi_- : F^{(1)} \cong (V_0)_-$  is injective, and it is surjective as well because  $F^{(1)}$  is a maximal negative definite subspace. Now we have a well defined linear map

$$(6.1.3) \quad X_{F^{(1)}} : (V_0)_- \rightarrow (V_0)_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}.$$

As  $\mathcal{F}^{(1)}$  is weakly compatible with  $E$ , the matrix of  $X_{F^{(1)}}$  relative to  $E$  has only finitely many nonzero entries, i.e.  $X_{F^{(1)}}$  is finitary. Further,  $\pi_- : F^{(1)} \cong (V_0)_-$  defines an  $\mathbb{R}$ -basis  $\{e''_i\}$  of  $F^{(1)}$  by  $\pi_-(e''_i) = e_i$ . Write  $e''_i = e_i + \sum_{j < 0} x_{j,i} e_j$ ; then  $(x_{j,i})$  is the matrix of  $X_{F^{(1)}}$ . The fact that  $F^{(1)}$  is  $b$ -negative definite, translates to the matrix condition  $I - {}^t(x_{j,i})(x_{j,i}) \gg 0$ , equivalently the operator condition  $I - {}^tX_{F^{(1)}}X_{F^{(1)}} \gg 0$ . Conversely if  $X : (V_0)_- \rightarrow (V_0)_+$  is finitary and satisfies  $I - {}^tX X \gg 0$ , then the real column span of its matrix relative to  $E$  is a maximal negative definite subspace  $F^{(1)}$ , and  $\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in D_0$ .

The same computation as in Section 4.1 shows that the block form matrices of elements of  $G_0$  act by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} X \\ I \end{pmatrix} \rightarrow \begin{pmatrix} AX+B \\ CX+D \end{pmatrix}$ , which has the same real column span as  $\begin{pmatrix} (AX+B)(CX+D)^{-1} \\ I \end{pmatrix}$ . So  $G_0$  acts by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \rightarrow (AX+B)(CX+D)^{-1}$ . In summary,

**Proposition 6.1.4.**  *$D_0$  is realized as the bounded domain of all finitary  $X : (V_0)_- \rightarrow (V_0)_+$  such that  $I - {}^tX X \gg 0$ , and there the action of  $G_0$  is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \rightarrow (AX+B)(CX+D)^{-1}$ .*

Again, there are  $q+1$  open  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},E}$  corresponding to nondegenerate signatures:

$$D_k = G_0(0, F_{(k)}, V_0) \text{ where } F_{(k)} = \text{Span}_{\mathbb{R}}\{e_{-k}, \dots, e_{-1}; e_{k+1}, \dots, e_q\} \text{ if } q < \infty, \\ F_{(k)} = \text{Span}_{\mathbb{R}}\{e_{-k}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ if } q = \infty,$$

More generally the  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},E}$  of signature  $(a, b, c) = (\text{pos}, \text{neg}, \text{nul})$  have  $a$  and  $c$  finite and  $\leq q$ . We denote them by

$$(6.1.5) \quad \begin{aligned} D_{a,b,c} &= G_0(0, (F_+ + F_- + F_0), V) \text{ where} \\ F_0 &= \text{Span}_{\mathbb{R}}\{e_{-c} + e_c, \dots, e_{-1} + e_1\} \text{ (null)} \\ F_+ &= \text{Span}_{\mathbb{R}}\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (positive)} \\ F_- &= \text{Span}_{\mathbb{R}}\{e_{c+1}, \dots, e_{c+b}\}, \quad q < \infty; \text{Span}_{\mathbb{R}}\{e_{c+1}, e_{c+2}, \dots\}, \quad q = \infty \text{ (negative)}. \end{aligned}$$

As in the complex case, the open orbits are the  $D_a = D_{a,b,0}$ ,  $a < \infty$  and  $a+b = q$ , i.e. the ones for  $c = 0$ . If  $q < \infty$  there is a unique closed orbit,  $D_{0,0,q} = \{(0, F^{(1)}, V_0) \in \mathfrak{X}_{\mathcal{F},E} \mid b(F^{(1)}, F^{(1)}) = 0\}$ ; it is in the closure of every orbit. If  $q = \infty$  there is no closed orbit.

The Cayley transforms are given by (4.1.5):  $c_k(e_j) = e_j$  if  $j \neq \pm k$  and, in the basis  $\{e_{-k}, e_k\}$  of  $\text{Span}_{\mathbb{R}}\{e_{-k}, e_k\}$ ,  $c_k$  has matrix  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . This sends real subspaces of  $V$  to real subspaces; that is why, in Section 4.1, we based (4.1.5) on the one variable Cayley transform that sends  $0 \rightarrow 1 \rightarrow \infty \rightarrow -1 \rightarrow 0$  and maps the unit disk to the right half plane. As a riemannian symmetric space, the real Grassmannian  $\mathfrak{X}_{\mathcal{F},E}$  has rank  $q$ . Just as in the complex case the  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},E}$  are the  $G_0(c_1 \dots c_s c_{s+1}^2 \dots c_{s+t}^2 \mathcal{F})$ , and the open ones are those for which

$s = 0$ . If  $q < \infty$  then  $G_0(c_1 \dots c_q \mathcal{F})$  is the closed orbit, and if  $q = \infty$  then there is no closed  $G_0$ -orbit on  $\mathfrak{X}_{\mathcal{F}, E}$ .

The maximal lim-compact subgroup of  $G_0$  is

$$\begin{aligned} K_0 &= SO(\infty) \times SO(q) = (\lim_{p \rightarrow \infty} SO(p)) \times SO(q) \text{ if } q < \infty, \\ K_0 &= SO(\infty) \times SO(\infty) = \lim_{p, q \rightarrow \infty} (SO(p) \times SO(q)) \text{ if } q = \infty. \end{aligned}$$

This corresponds to the  $b$ -orthogonal decomposition  $\mathbb{R}^{\infty, q} = (V_0)_+ \oplus (V_0)_-$ . Let  $\mathcal{F} = (F_k)$  be a generalized flag in  $V = \mathbb{R}^{\infty, q}$  that is weakly compatible with  $E$ . Let  $\mathcal{F}^{(1)} \in \mathfrak{X}_{\mathcal{F}, E}$  so that  $D = G_0(\mathcal{F}^{(1)})$  is an open  $G_0$ -orbit. Then we may assume that  $\mathcal{F}^{(1)}$  is compatible with our choice of  $E$ , so it fits the decomposition  $\mathbb{R}^{\infty, q} = (V_0)_+ \oplus (V_0)_-$  in the sense that

$$(6.1.6) \quad \mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap (V_0)_+) \oplus (F_k^{(1)} \cap (V_0)_-).$$

Then  $K_0(\mathcal{F}^{(1)})$  is the real analog – in fact a real form – of the base cycle in the complexification of  $D$ . Concretely,  $K_0(\mathcal{F}^{(1)})$  is the product of “smaller” real flag manifolds,

$$(6.1.7) \quad \begin{aligned} Y &= Y_1 \times Y_2 \text{ where} \\ Y_1 &= K_0(\mathcal{F}^{(1)} \cap (V_0)_+) = SO(\infty)(\mathcal{F}^{(1)} \cap (V_0)_+) \text{ in } (V_0)_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap (V_0)_-) = SO(q)(\mathcal{F}^{(1)} \cap (V_0)_-) \text{ in } (V_0)_-. \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}^{(1)} \cap (V_0)_+ &= ((F_1^{(1)} \cap (V_0)_+) \subset \dots \subset (F_n^{(1)} \cap (V_0)_+)), \\ \mathcal{F}^{(1)} \cap (V_0)_- &= ((F_1^{(1)} \cap (V_0)_-) \subset \dots \subset (F_n^{(1)} \cap (V_0)_-)). \end{aligned}$$

The signature sequence  $\{(a_k, b_k)\}$ , where  $h$  has signature  $(a_k, b_k, 0)$  on  $F_k^{(1)}$ , specifies the open orbit in  $\mathfrak{X}_{\mathcal{F}, E}$  and the factors of  $Y$ .

This shows that the  $G$ -translates of  $Y$  contained in  $D$  correspond to the decompositions  $V_0 = W'_0 \oplus W''_0$  where (i)  $W'_0$  is a maximal positive definite subspace such that  $(V_0)_+ \cap W'_0$  has finite codimension in both  $W'_0$  and  $(V_0)_+$ , and (ii)  $W''_0$  is a maximal negative definite subspace such that  $(V_0)_- \cap W''_0$  has finite codimension in both  $W''_0$  and  $(V_0)_-$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_0)_+$  the correspondence depends only on  $W'_0$ , and if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_0)_-$  it depends only on  $W''_0$ . Any two such decompositions  $V_0 = W'_0 \oplus W''_0$  are  $G$ -equivalent.

**Definition 6.1.8.** The *positive real bounded symmetric domain*  $\mathcal{B}_E^+$  associated to  $(V_0, b, E)$  is the space of all maximal positive definite subspaces  $W'_0 \subset V_0$  such that  $W'_0 \cap (V_0)_+$  has finite codimension in both  $W'_0$  and  $(V_0)_+$ . The *negative bounded symmetric domain*  $\mathcal{B}_E^-$  associated to  $(V_0, b, E)$  is the space of all maximal negative definite subspaces  $W''_0 \subset V_0$  such that  $W''_0 \cap (V_0)_-$  has finite codimension in both  $W''_0$  and  $(V_0)_-$ .

As constructed, each element  $W'_0 \in \mathcal{B}_E^+$  is in the  $G$ -orbit of  $(V_0)_+$ . Relative to the basis  $E$  we look at  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$  such that  $gW'_0 \in \mathcal{B}_E^+$ , in other words such that the column span of  $\begin{pmatrix} A \\ C \end{pmatrix}$  is positive definite. The column span is preserved under right multiplication by  $A$ , so the positive definite condition is  ${}^t \begin{pmatrix} I \\ -CA^{-1} \end{pmatrix} \cdot \begin{pmatrix} I \\ CA^{-1} \end{pmatrix} \gg 0$ . In other words  $gW'_0 \in \mathcal{B}_E^+$  simply means that  $gW'_0$  is the column span of an infinite real matrix  $\begin{pmatrix} I \\ X_1 \end{pmatrix}$  such that  $I - {}^t X_1 X_1 \gg 0$ . Similarly  $gW''_0 \in \mathcal{B}_E^-$  simply means that  $gW''_0$  is the column span of an infinite real matrix  $\begin{pmatrix} X_2 \\ I \end{pmatrix}$  such that  $I - {}^t X_2 X_2 \gg 0$ . The  $G$ -stabilizer of  $0 \in \mathcal{B}_E^+$  is the parabolic  $P$  consisting of all  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ , while the  $G$ -stabilizer of  $0 \in \mathcal{B}_E^-$  is the opposite parabolic  ${}^t P = P^{opp}$  consisting of all  $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ . Reformulating this,

**Lemma 6.1.9.** *Suppose that  $G_0 = SO(\infty, q)$ ,  $q \leq \infty$ . Then the real positive bounded symmetric domain associated to  $(V, G_0, E)$  is  $\mathcal{B}_E^+ \cong \{X_1 \in \mathbb{R}^{\infty \times q} \mid I - {}^t X_1 X_1 \gg 0\}$  in  $G/P$ , and the negative real bounded symmetric domain for  $(V, G_0, E)$  is  $\mathcal{B}_E^- \cong \{X_2 \in \mathbb{R}^{q \times \infty} \mid I - {}^t X_2 X_2 \gg 0\}$  in  $G/P^{opp}$ .*

The action of  $G_0$  on these bounded symmetric domains is linear fractional, as described in Section 4.1 for the complex case. The proof of Theorem 5.2.6 is valid here, giving us the following structure theorem.

**Theorem 6.1.10.** *Let  $G_0 = SO(\infty, q)$  with  $2 < q \leq \infty$ . Let  $D$  be an open  $G_0$ -orbit  $G((0, F^{(1)}, V_0))$  in the real flag manifold  $\mathfrak{X}_{\mathcal{F}, E}$ . Then the positive definite bounded symmetric domain  $\mathcal{B}_E^+$  for  $(V, G_0, E)$  is the set of all positive definite  $G$ -translates of  $(V_0)_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_E^-$  for  $(V, G_0, E)$  is the set of all negative definite  $G$ -translates of  $(V_0)_-$ . The  $\mathcal{B}_E^\pm$  are diffeomorphic. There are three cases for the structure of the cycle space, as follows.*

*If every space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_E^+$ .*

*If every space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_E^-$ .*

*If some space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_E^+ \times \mathcal{B}_E^-$ .*

## 6.2 The Quaternionic Bounded Symmetric Domain for $Sp(\infty, q)$ .

We now look at the quaternionic analog of Section 6.1. For that, we consider a quaternionic vector space  $V_{\mathbb{H}} = \mathbb{H}^{\infty, q}$ , one of whose underlying complex structures is that of  $V = \mathbb{C}^{\infty, 2q}$ . We look at the bounded symmetric domain of maximal negative definite quaternionic subspaces of  $(V_{\mathbb{H}}, h)$ . As suggested by Section 3.3, the complex basis  $E$  of  $V$  is replaced by an  $\mathbb{H}$ -basis

$$(6.2.1) \quad \begin{aligned} L &= \{\dots, v_{-2}, v_{-1}; v_1, v_2, \dots, v_q\} \text{ for } q < \infty, \\ L &= \{\dots, v_{-2}, v_{-1}; v_1, v_2, v_3, \dots\} \text{ for } q = \infty. \end{aligned}$$

The relation with  $E$  is  $v_i = e_{2i}$  for  $i < 0$  and  $v_j = e_{2j-1}$  for  $j > 0$ . The  $\mathbb{H}$ -hermitian form  $h$  is defined by  $h(v_i, v_j) = \delta_{i,j}$  for  $i < 0$  and  $h(v_i, v_j) = -\delta_{i,j}$  for  $i > 0$ .

The real group is  $G_0 = Sp(\infty, q)$ ,  $q \leq \infty$ . We view  $G_0$  as a closed subgroup of the quaternionic linear group  $G := SL(\infty + q; \mathbb{H})$ . The flag is  $\mathcal{F} = \{F\}$  where  $F = \text{Span}_{\mathbb{H}}\{e_i \mid i > 0\}$ . That gives us the *quaternionic* flag manifold

$$(6.2.2) \quad \mathfrak{X}_{\mathcal{F}, L} = \{\text{subspaces } F^{(1)} \subset V_{\mathbb{H}} \mid (0, F^{(1)}, V_{\mathbb{H}}) \text{ is } L\text{-commensurable to } \mathcal{F}\} = G(\mathcal{F})$$

where the second equality follows as in the argument of Lemma 2.2.3. Note that  $\mathfrak{X}_{\mathcal{F}, L}$  is a quaternionic Grassmann manifold. The domain of interest to us in this context is

$$(6.2.3) \quad D_0 = \{(0, F^{(1)}, V_{\mathbb{H}}) \in \mathfrak{X}_{\mathcal{F}, L} \mid F^{(1)} \text{ is a maximal } h\text{-negative definite subspace of } V_{\mathbb{H}}\}.$$

Now consider the  $h$ -orthogonal decomposition  $V_{\mathbb{H}} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$  where  $(V_{\mathbb{H}})_+$  denotes  $\text{Span}_{\mathbb{H}}\{e_i \mid i < 0\}$  and  $(V_{\mathbb{H}})_-$  denotes  $\text{Span}_{\mathbb{H}}\{e_i \mid i > 0\}$ . Consider the corresponding orthogonal projections  $\pi_+ : V_{\mathbb{H}} \rightarrow (V_{\mathbb{H}})_+$  and  $\pi_- : V_{\mathbb{H}} \rightarrow (V_{\mathbb{H}})_-$ . The kernel of  $\pi_-$  is  $h$ -positive definite so it has zero intersection with  $F^{(1)}$  for any  $\mathcal{F}^{(1)} = (0, F^{(1)}, V_{\mathbb{H}}) \in D_0$ . Thus  $\pi_- : F^{(1)} \cong (V_{\mathbb{H}})_-$  is injective. Since  $F^{(1)}$  is a maximal  $h$ -negative definite subspace  $\pi_- : F^{(1)} \cong (V_{\mathbb{H}})_-$  is surjective as well. Now we have a well defined  $\mathbb{H}$ -linear map

$$(6.2.4) \quad X_{F^{(1)}} : (V_{\mathbb{H}})_- \rightarrow (V_{\mathbb{H}})_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}.$$

As  $\mathcal{F}^{(1)}$  is weakly compatible with  $L$ , the matrix of  $X_{F^{(1)}}$  relative to  $L$  has only finitely many nonzero entries, i.e.  $X_{F^{(1)}}$  is finitary. Using  $\pi_- : F^{(1)} \cong (V_{\mathbb{H}})_-$  defines an  $\mathbb{H}$ -basis  $\{v''_i\}$  of  $F^{(1)}$

by  $\pi_-(v_i'') = v_i$ . Write  $v_i'' = v_i + \sum_{j < 0} x_{j,i} v_j$ ; then  $(x_{j,i})$  is the matrix of  $X_{F^{(1)}}$ . The fact that  $F^{(1)}$  is  $h$ -negative definite, translates to the matrix condition  $I - (x_{j,i})^* (x_{j,i}) \gg 0$ , equivalently the operator condition  $I - X_{F^{(1)}}^* X_{F^{(1)}} \gg 0$ . Conversely if  $X : (V_{\mathbb{H}})_- \rightarrow (V_{\mathbb{H}})_+$  is finitary and satisfies  $I - X^* X \gg 0$ , then the quaternionic column span of its matrix relative to  $L$  is a maximal negative definite subspace  $F^{(1)}$ , and  $\mathcal{F}^{(1)} = (0, F^{(1)}, V_{\mathbb{H}}) \in D_0$ .

The same computation as in Section 4.1 shows that the block form matrices of elements of  $G_0$  act by  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} X \\ I \end{pmatrix} \rightarrow \begin{pmatrix} AX+B \\ CX+D \end{pmatrix}$ , which has the same quaternionic column span as  $\begin{pmatrix} (AX+B)(CX+D)^{-1} \\ I \end{pmatrix}$ . So  $G_0$  acts by the linear fractional  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \rightarrow (AX+B)(CX+D)^{-1}$ . In summary,

**Proposition 6.2.5.**  *$D_0$  is realized as the bounded domain of all finitary  $X : (V_{\mathbb{H}})_- \rightarrow (V_{\mathbb{H}})_+$  such that  $I - X^* X \gg 0$ , and there the action of  $G_0$  is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \rightarrow (AX+B)(CX+D)^{-1}$ .*

Again, there are  $q+1$  open  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},L}$  corresponding to nondegenerate signatures:

$$\begin{aligned} D_k &= G_0(0, F_{(k)}, V_{\mathbb{H}}) \text{ where } F_{(k)} = \text{Span}_{\mathbb{H}}\{v_{-k}, \dots, v_{-1}; v_{k+1}, \dots, v_q\} \text{ if } q < \infty, \\ &F_{(k)} = \text{Span}_{\mathbb{H}}\{v_{-k}, \dots, v_{-1}; v_{k+1}, v_{k+2}, \dots\} \text{ if } q = \infty, \end{aligned}$$

More generally the  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},L}$  of signature  $(a, b, c) = (\text{pos}, \text{neg}, \text{nul})$  have  $a$  and  $c$  finite and  $\leq q$ . We denote them by

$$\begin{aligned} (6.2.6) \quad D_{a,b,c} &= G_0(0, (F_+ + F_- + F_0), V_{\mathbb{H}}) \text{ where} \\ &F_0 = \text{Span}_{\mathbb{H}}\{v_{-c} + v_c, \dots, v_{-1} + v_1\} \text{ (null)} \\ &F_+ = \text{Span}_{\mathbb{H}}\{v_{-c-a}, \dots, v_{-c-1}\} \text{ (positive)} \\ &F_- = \text{Span}_{\mathbb{H}}\{v_{c+1}, \dots, v_{c+b}\} \text{ if } q < \infty, \\ &\text{Span}_{\mathbb{H}}\{v_{c+1}, v_{c+2}, \dots\} \text{ if } q = \infty \text{ (negative).} \end{aligned}$$

The open orbits are the  $D_a = D_{a,b,0}$ ,  $a < \infty$  and  $a+b = q$ , i.e. the ones for  $c = 0$ . If  $q < \infty$  there is a unique closed orbit,  $D_{0,0,q} = \{(0, F^{(1)}, V_{\mathbb{H}}) \in \mathfrak{X}_{\mathcal{F},L} \mid h(F^{(1)}, F^{(1)}) = 0\}$ ; it is in the closure of every orbit. If  $q = \infty$  there is no closed orbit.

The Cayley transforms are given by (4.1.5):  $c_k(v_j) = v_j$  if  $j \neq \pm k$  and, in the basis  $\{v_{-k}, v_k\}$  of  $\text{Span}_{\mathbb{H}}\{v_{-k}, v_k\}$ ,  $c_k$  has matrix  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . This sends quaternionic subspaces of  $V$  to quaternionic subspaces. As a riemannian symmetric space, the quaternion Grassmannian  $\mathfrak{X}_{\mathcal{F},L}$  has rank  $q$ . Just as in the complex case the  $G_0$ -orbits on  $\mathfrak{X}_{\mathcal{F},L}$  are the  $G_0(c_1 \dots c_s c_{s+1}^2 \dots c_{s+t}^2 \mathcal{F})$ , and the open ones are those for which  $s = 0$ . If  $q < \infty$  then  $G_0(c_1 \dots c_q \mathcal{F})$  is the closed orbit, and if  $q = \infty$  then there is no closed  $G_0$ -orbit on  $\mathfrak{X}_{\mathcal{F},L}$ .

The maximal lim-compact subgroup of  $G_0$  is

$$\begin{aligned} K_0 &= Sp(\infty) \times Sp(q) = (\lim_{p \rightarrow \infty} Sp(p)) \times Sp(q) \text{ if } q < \infty, \\ K_0 &= Sp(\infty) \times Sp(\infty) = \lim_{p,q \rightarrow \infty} (Sp(p) \times Sp(q)) \text{ if } q = \infty. \end{aligned}$$

This corresponds to the  $h$ -orthogonal decomposition  $\mathbb{H}^{\infty,q} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$ . Let  $\mathcal{F} = (F_k)$  be a generalized flag in  $V = \mathbb{H}^{\infty,q}$  that is weakly compatible with  $L$ . Let  $\mathcal{F}^{(1)} \in \mathfrak{X}_{\mathcal{F},L}$  so that  $D = G_0(\mathcal{F}^{(1)})$  is an open  $G_0$ -orbit. Then we may assume that  $\mathcal{F}^{(1)}$  is compatible with our choice of  $L$ , so it fits the decomposition  $\mathbb{H}^{\infty,q} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$  in the sense that

$$(6.2.7) \quad \mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap (V_{\mathbb{H}})_+) \oplus (F_k^{(1)} \cap (V_{\mathbb{H}})_-).$$

Then  $K_0(\mathcal{F}^{(1)})$  is the quaternionic analog – in fact a quaternion form – of the base cycle when the latter is viewed as a quaternionic manifold. Concretely,  $K_0(\mathcal{F}^{(1)})$  is the product of “smaller”

quaternionic flag manifolds,

$$Y = Y_1 \times Y_2 \text{ where}$$

$$(6.2.8) \quad \begin{aligned} Y_1 &= K_0(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+) = Sp(\infty)(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+) \text{ in } (V_{\mathbb{H}})_+ \text{ and} \\ Y_2 &= K_0(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-) = Sp(q)(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-) \text{ in } (V_{\mathbb{H}})_- \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+ &= ((F_1^{(1)} \cap (V_{\mathbb{H}})_+) \subset \cdots \subset (F_n^{(1)} \cap (V_{\mathbb{H}})_+)), \\ \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_- &= ((F_1^{(1)} \cap (V_{\mathbb{H}})_-) \subset \cdots \subset (F_n^{(1)} \cap (V_{\mathbb{H}})_-)). \end{aligned}$$

The signature sequence  $\{(a_k, b_k)\}$ , where  $h$  has signature  $(a_k, b_k, 0)$  on  $F_k^{(1)}$ , specifies the open orbit in  $\mathfrak{X}_{\mathcal{F}, L}$  and the factors of  $Y$ .

This shows that the  $G$ -translates of  $Y$  contained in  $D$  correspond to the decompositions  $V_H = W'_{\mathbb{H}} \oplus W''_{\mathbb{H}}$  where (i)  $W'_{\mathbb{H}}$  is a maximal positive definite  $\mathbb{H}$ -subspace such that  $(V_{\mathbb{H}})_+ \cap W'_{\mathbb{H}}$  has finite codimension in both  $(V_{\mathbb{H}})_+$  and  $\cap W'_{\mathbb{H}}$ , and (ii)  $W''_{\mathbb{H}}$  is a maximal negative definite subspace such that  $(V_{\mathbb{H}})_- \cap W''_{\mathbb{H}}$  has finite codimension in both  $(V_{\mathbb{H}})_-$  and  $W''_{\mathbb{H}}$ . If  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+$  the correspondence depends only on  $W'_{\mathbb{H}}$ , and if  $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-$  it depends only on  $W''_{\mathbb{H}}$ . Any two such decompositions  $V_{\mathbb{H}} = W'_{\mathbb{H}} \oplus W''_{\mathbb{H}}$  are  $G$ -equivalent.

**Definition 6.2.9.** The positive quaternionic bounded symmetric domain  $\mathcal{B}_L^+$  associated to  $(V_{\mathbb{H}}, b, L)$  is the space of all maximal positive definite subspaces  $W'_{\mathbb{H}} \subset V_{\mathbb{H}}$  such that  $W'_{\mathbb{H}} \cap (V_{\mathbb{H}})_+$  has finite codimension in both  $W'_{\mathbb{H}}$  and  $(V_{\mathbb{H}})_+$ . The negative quaternionic bounded symmetric domain  $\mathcal{B}_L^-$  associated to  $(V_{\mathbb{H}}, b, L)$  is the space of all maximal negative definite subspaces  $W''_{\mathbb{H}} \subset V_{\mathbb{H}}$  such that  $W''_{\mathbb{H}} \cap (V_{\mathbb{H}})_-$  has finite codimension in both  $W''_{\mathbb{H}}$  and  $(V_{\mathbb{H}})_-$ .

As constructed, each element  $W'_{\mathbb{H}} \in \mathcal{B}_L^+$  is in the  $G$ -orbit of  $(V_{\mathbb{H}})_+$ . Relative to the basis  $L$  we look at  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$  such that  $gW'_{\mathbb{H}} \in \mathcal{B}_L^+$ , in other words such that the column span of  $\begin{pmatrix} A \\ C \end{pmatrix}$  is positive definite. The column span is preserved under right multiplication by  $A$ , so the positive definite condition is  ${}^t \begin{pmatrix} I \\ -CA^{-1} \end{pmatrix} \cdot \begin{pmatrix} I \\ CA^{-1} \end{pmatrix} \gg 0$ . In other words  $gW'_{\mathbb{H}} \in \mathcal{B}_L^+$  simply means that  $gW'_{\mathbb{H}}$  is the column span of an infinite matrix  $\begin{pmatrix} I \\ X_1 \end{pmatrix}$  such that  $I - {}^tX_1X_1 \gg 0$ . Similarly  $gW''_{\mathbb{H}} \in \mathcal{B}_L^-$  simply means that  $gW''_{\mathbb{H}}$  is the column span of an infinite matrix  $\begin{pmatrix} X_2 \\ I \end{pmatrix}$  such that  $I - {}^tX_2X_2 \gg 0$ . The  $G$ -stabilizer of  $0 \in \mathcal{B}_L^+$  is the parabolic  $P$  consisting of all  $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ , while the  $G$ -stabilizer of  $0 \in \mathcal{B}_L^-$  is the opposite parabolic  ${}^tP = P^{opp}$  consisting of all  $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ . Reformulating this,

**Lemma 6.2.10.** Suppose that  $G_0 = Sp(\infty, q)$ ,  $q \leq \infty$ . Then the quaternionic positive bounded symmetric domain associated to the triple  $(V, G_0, L)$  is  $\mathcal{B}_L^+ \cong \{X_1 \in \mathbb{H}^{\infty \times q} \mid I - {}^tX_1X_1 \gg 0\}$  in  $G/P$ , and the corresponding quaternionic negative bounded symmetric domain for  $(V, G_0, L)$  is  $\mathcal{B}_L^- \cong \{X_2 \in \mathbb{H}^{q \times \infty} \mid I - {}^tX_2X_2 \gg 0\}$  in  $G/P^{opp}$ .

The action of  $G_0$  on these bounded symmetric domains is linear fractional, as described in Section 4.1 for the complex case. The proof of Theorem 5.2.6 is valid here, giving us the following structure theorem.

**Theorem 6.2.11.** Let  $G_0 = Sp(\infty, q)$  with  $q \leq \infty$ . Let  $D$  be an open  $G_0$ -orbit  $G(\mathcal{F}^{(1)})$  in the quaternionic flag manifold  $\mathfrak{X}_{\mathcal{F}, L}$ . Then the positive definite bounded symmetric domain all positive definite  $G$ -translates of  $(V_{\mathbb{H}})_+$  and the negative definite bounded symmetric domain  $\mathcal{B}_L^-$  for  $(V, G_0, L)$  is the set of all negative definite  $G$ -translates of  $(V_{\mathbb{H}})_-$ . The  $\mathcal{B}_L^{\pm}$  are diffeomorphic. There are three cases for the structure of the cycle space, as follows.

If every space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is positive definite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_L^+$ .

If every space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is negative definite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_L^-$ .

If some space  $F_k^{(1)} \in \mathcal{F}^{(1)}$  is indefinite then  $\mathcal{M}_D$  is diffeomorphic to  $\mathcal{B}_L^+ \times \mathcal{B}_L^-$ .

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